Mallows-Smoothed Distribution over Rankings Approach for Modeling Choice

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Assortment optimization is an important problem that arises in many applications including retailing and online advertising. The goal in such problems is to determine a revenue/profit maximizing subset of products to offer from a large universe of products when customers exhibit stochastic substitution behavior. We consider a mixture of Mallows model for demand, which can be viewed as a “smoothed” generalization of the class of sparse rank-based choice models, designed to overcome some of its key limitations. In spite of these advantages, the Mallows distribution has an exponential support size and does not admit a closed-form expression for choice probabilities.

We first conduct a case study using a publicly available data set involving real-world preferences on sushi types to show the benefits of Mallows-based smoothing. We show that smoothing significantly improves both the prediction and the decision accuracy on this data set. We then present an efficient procedure to compute the choice probabilities for any assortment under the mixture of Mallows model. Surprisingly, this finding allows to formulate a compact mixed integer program (MIP) that leads to a practical approach for solving the constrained assortment optimization problem under a general mixture of Mallows model. To complement this MIP formulation, we also exploit additional structural properties of the underlying distribution to propose several polynomial-time approximation schemes. These are the first efficient algorithms with provably near-optimal performance guarantees for the assortment optimization problem under the Mallows or the mixture of Mallows model in such generality.

1. Introduction

What subset (or assortment) of products to offer to customers is a key decision problem arising in many application contexts. A concrete application is that of a retailer who carries a large universe of products but can offer only a subset of the products in each store, either online or offline. The retailer typically chooses the offer set to maximize the expected revenue/profit earned from each arriving customer. The retailer might also want to maximize conversion rate, which is equal to the revenue/profit with all product prices set to one.

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To make the assortment decision, the retailer needs a model to capture the stochastic demand. This demand model must account for *product substitution* behavior, whereby customers may substitute to an available product (say, a dark blue shirt) when their most preferred product (say, a black one) is not offered. The substitution behavior makes the demand for each offered product a function of the *entire* offer set, increasing the complexity of the demand model. Nevertheless, existing work has shown that demand models where substitution effects are incorporated provide significantly more accurate predictions than those that overlook such effects (Farias et al. 2013, van Ryzin and Vulcano 2014). The common approach to capturing substitution is through a discrete *choice model* that specifies the demand as the probability $P(a | S)$ of a random customer choosing product $a$ from offer set $S$.

In this work, we assume that the choice model is specified by a distribution over all possible rankings/preference lists of the products. Each preference list specifies a rank ordering of the products, with lower ranked products being more preferred. In each choice instance, a customer samples a preference list from the underlying distribution and then chooses the most preferred available product (possibly the no-purchase option) from her list. Modeling customer preferences using a distribution over all rankings has recently received a great deal of attention (Farias et al. 2013, van Ryzin and Vulcano 2014, Honhon et al. 2012, Jagabathula and Rusmevichientong 2017, 2018, Aouad et al. 2015). This is indeed a very rich framework as it can potentially accommodate distributions with exponentially large support sizes and, therefore, can capture complex substitution patterns. However, available data are usually insufficient to identify such a complex model. Therefore, ‘sparsity’, measured as the support size, is employed as a selection criterion to pick a model from the set of models consistent with the data. Specifically, it is assumed that the distribution has a support size $K$, which is polynomial in the number of products $n$. Sparsity results in data-driven model selection (Farias et al. 2013, Jagabathula and Rusmevichientong 2017, van Ryzin and Vulcano 2017), obviating the need for imposing arbitrary parametric structures. In fact, such sparse models are a special case of the mixture of Mallows model. Indeed, as the concentration parameters $\theta_k$ tend to infinity for all $k$, the distribution concentrates around each of the $K$ modes, yielding the sparse rank-based model.

Despite their generality, sparse rank-based models cannot account for “noise” or deviations from the $K$ ranked-lists in the support and tend to over-fit to the observed data. This shortcoming limits their modeling flexibility, resulting in potentially unrealistic predictions. Because $K \ll n!$, the sparse rank-based model assigns a zero chance to any event where a customer ranks the $n$ underlying products even slightly different from any of the $K$ rankings in the support. Furthermore, when $K < n$, the sparse model assigns a zero probability of observing certain choices. However, in real (holdout) data, one may observe choices that are inconsistent with any of the $K$ rankings, making the model predictions unrealistic. Another issue with the sparse model is potential over-fit when the observed choices lack sufficient variation in the offer sets. For instance, if we only observe market shares, or
choice probabilities under the complete assortment, then only the top-ranked products in each of the preference lists are identified. As a result, this model cannot generalize choice predictions to other offer sets; for instance, we don’t know how a customer with a particular preference list will choose if her most preferred product is stocked out.

To address these issues, we generalize the sparse rank-based models by “smoothing” them using the Mallows kernel. We assume that the underlying probability distribution is specified by a mixture of \( K \) Mallows models and focus on assortment optimization under this generalized class. The Mallows distribution was introduced in the mid-1950’s (Mallows 1957), and has since been the most popular member of the so-called distance-based ranking models, which are characterized by a modal ranking \( \omega \) and a concentration parameter \( \theta \). As formally explained in Section 2, the probability that a ranking \( \sigma \) is sampled falls exponentially as \( e^{-\theta \cdot d(\sigma, \omega)} \), where \( d(\cdot, \cdot) \) is the distance between \( \sigma \) and \( \omega \). Clearly, different distance functions result in different models. The Mallows model utilizes the Kendall-Tau distance function, which measures distance as the number of pairwise disagreements between the two rankings. Intuitively, the Mallows model assumes that consumer preferences are concentrated around a central permutation, with the likelihood of large deviations being low. More generally, the mixture of Mallows model with \( K \) segments is specified by the modal rankings \( \omega_1, \ldots, \omega_K \), concentration parameters \( \theta_1, \ldots, \theta_K \) and probabilities \( p_1, \ldots, p_K \), where \( p_k \) specifies the probability that a random customer belongs to Mallows segment \( k \) with modal ranking \( \omega_k \) and concentration parameter \( \theta_k \).

The mixture of Mallows model is a natural alternative to sparse rank-based models, allowing for deviations from the modal rankings and assigning a non-zero probability to every possible choice. In fact, as shown in Section 3, smoothing can significantly improve the parsimony of sparse rank-based models, with two to three mixture components providing more accurate choice predictions than a rank-based model with support size 1,250. See Section 3 for a discussion on how smoothing addresses the limitations of sparse rank-based models.

1.1. Our contributions

Our main contributions in this paper include: (a) A detailed case study showcasing the benefits of smoothing; (b) Efficient computation of the choice probabilities under a mixture of Mallows model and, hence, the expected revenue/profit, for any offer set; (c) A compact mixed integer program (MIP) for finding the optimal assortment under a general set of constraints; and (d) Efficient algorithms for identifying provably near-optimal assortments. These contribution can be succinctly summarizes as follows. 

**Case study.** In Section 3, we conduct a computational study using a publicly available sushi data set (Kamishima et al. 2005) to illustrate the benefits of Mallows-based smoothing. We find that
smoothing results in significant improvements (up to 15% when using the KL divergence metric) in the accuracy of predicting pairwise choice probabilities, i.e., choice probabilities of the form $P(a|S)$ where $|S| = 2$. In this case, the choice probability represents the likelihood of preferring one sushi variety over another. Smoothing also improves the decision accuracy and leads to up to 2% improvement in revenue. Through our case study, we also tackle the problem of estimating the parameters of the mixture model from choice observations. Our approach involves using existing techniques for fitting a sparse rank-based model from choice data and then smoothing it, resulting in a Mallows-smoothed distribution. We note that, while there is a large body of work in the machine learning literature for estimating the parameters of a Mallows model (Lu and Boutilier 2014, Awasthi et al. 2014, Dwork et al. 2001, Lebanon and Mao 2008), these papers rely on observations such as top/bottom-k ranked items, complete rankings, and tied rankings, and not on choice observations. These techniques do not extend to choice data. We also show that using a mixture of Mallows model strongly outperforms (both on prediction and decision accuracy) the mixture of MNL models, which is another very popular choice model. Having motivated the benefits of smoothing, we then turn to the fundamental challenges of computing choice probabilities and identifying optimal assortments.

**Exact computation of choice probabilities.** In Section 4, we present an efficient procedure to compute the exact choice probabilities $P(a|S)$ under a general mixture of Mallows model. Since the distribution being considered has an exponential support size, computing the choice probabilities for a fixed offer set $S$ by definition requires marginalizing the distribution by summing it over an exponential number of rankings, and is therefore a non-trivial computational task. In fact, computing the probability of a general partial order under the Mallows distribution is known to be a $\#P$-hard problem (Lu and Boutilier 2014, Brightwell and Winkler 1991). Currently, the only class of partial orders whose probabilities are known to be efficiently computable is the class of partitioned preferences (Lebanon and Mao 2008); while this class includes top-k/bottom-k ranked lists, it does not include other popular partial orders such as pairwise comparisons and choice observations.

To address this challenge, we show that the choice probabilities for an offer set under the Mallows model can be expressed as the unique solution to a system of linear equations that can be solved in $O(n^3)$ time. To obtain this characterization, we exploit the repeated insertion method (RIM) introduced by Doignon et al. (2004) for sampling rankings according to the Mallows distribution. We also provide an efficiently computable *closed-form expression* for the choice probabilities for a given offer set under the mixture of Mallows distribution. In particular, we prove that these choice probabilities are expressible through a discrete convolution, and therefore, can be computed using the fast Fourier transform. This provides an alternative approach to efficiently compute the choice probabilities.
A mixed integer programming formulation. Building on our procedure for computing the choice probabilities, we present in Section 5 a compact mixed integer linear program (MIP) with $O(Kn^3)$ variables, $O(n)$ binary variables, and $O(Kn^3)$ constraints for the assortment optimization problem under a general mixture of Mallows model with $K$ segments. Our formulation does not require any technical assumption. Furthermore, by exploiting certain structural properties of the distribution, we obtain a stronger formulation with $O(n^2)$ variables and $O(n)$ binary variables and also implement a variable bound strengthening. In Section 6, we conduct computational experiments using synthetic data, showing that the computation time of our MIP scales well to large problem sizes. We note that our MIP also offers a practical approach to incorporate additional business constraints in determining the optimal assortment.

Approximation schemes for assortment optimization. To complement our MIP formulation, which computes an optimal assortment but does not come with any theoretical guarantees in terms of running time, we present in Section 7 two approximation schemes for efficiently approaching optimal revenues within any degree of accuracy. These are the first provably near-optimal algorithms for assortment optimization under the mixture of Mallows model, even with a single segment.

- We first present a polynomial-time approximation scheme (PTAS) for a broad class of constrained assortment optimization problems, including those with cardinality, knapsack, or matroid constraints. This particular PTAS holds under the assumption that the no-purchase option is ranked last in the modal rankings of all Mallows segments in the mixture. For any $\epsilon > 0$, our algorithm computes in $O(Kn^{O(1/\epsilon)})$ time an assortment whose expected revenue is within factor $1 - \epsilon$ of optimal. This PTAS is based on establishing a surprising sparsity property, stating that, under the above-mentioned assumption, there exists a near-optimal assortment consisting of only $O(1/\epsilon)$ products. Therefore, enumerating over all such subsets results in a $(1 - \epsilon)$-approximation to the constrained assortment optimization problem.

- Our most general algorithmic result consists of a quasi-PTAS for the assortment optimization problem, without the need for any technical assumptions. Our algorithm exploits a multitude of monotonicity properties inherent to optimal assortments and their robustness to certain structural alterations. We remind the reader that an algorithm is said to be a quasi-PTAS if, for any accuracy level $\epsilon > 0$, it computes an assortment with an expected revenue of at least $1 - \epsilon$ times the optimal revenue, running in $O(n^{\text{polylog}(n)})$ time, potentially with an exponential dependency on $1/\epsilon$. In our case, we also incur an exponential dependency on the number of segments $K$ and $\log \Delta$, where $\Delta = \frac{r_{\max}}{r_{\min}}$ is the ratio between the maximum and minimum prices. More precisely, the running time of our algorithm is $O_{\epsilon}(n^{O((1/\epsilon)O(K))}K^{2, \log_{1+\epsilon} O(K) (n\Delta)})$ where to avoid cumbersome notation, we use $O_{\epsilon}()$ to suppress polynomial dependencies on $1/\epsilon$, meaning that $O_{\epsilon}(f(n)) = O(\text{poly}(1/\epsilon) \cdot f(n))$. From a
running time perspective, due to the exponential dependency on $K$ and $\log \Delta$, we indeed attain a quasi-PTAS for a fixed number of segments $K$, as long as $\Delta = \text{poly}(n)$. At this level of generality, exponential dependence on $K$ cannot be improved. To see this, consider the sparse rank-based model, which is a special case of the Mallows mixture model. Aouad et al. (2018) show that when the support size $K = n$, it is impossible to obtain an approximation to even within $\Omega(1/n^{1-\epsilon})$ of the optimal revenue in $\text{poly}(n, K)$-time, unless $P = NP$. Therefore, exponential dependence on $K$ is necessary in the absence of other structural assumptions.

1.2. Directly-related literature

A large number of parametric models over rankings have extensively been studied in the areas of statistics, transportation, marketing, economics, and operations management (see Marden (1995) for a detailed survey of most of these models). Particularly, our work has connections to several lines of research in machine learning and operations management.

The existing work in machine learning has focused on designing computationally efficient algorithms for estimating the model parameters from commonly available observations, such as complete rankings, top-$k$/bottom-$k$ lists, pairwise comparisons, etc. The developed techniques mainly consist of efficient algorithms for computing the likelihood of the observed data (Lebanon and Mao 2008, Guiver and Snelson 2009) and techniques for sampling from the distributions conditioned on observed data (Lu and Boutilier 2014, Meila et al. 2012). The Plackett-Luce (PL) model, the Mallows model, and their variants have been by far the most studied models in this literature.

On the other hand, the work in operations management has mainly focused on designing optimization algorithms to efficiently find optimal or near-optimal subsets of products. Several parametric choice models from the Random Utility Maximization (RUM) class have extensively been studied in diverse areas, including marketing, transportation, economics, and operations management. Among these models, the Multinomial Logit (MNL) model has been the most commonly studied in this literature. The MNL model was made popular by the work of McFadden (1978) and has been shown by Yellott (1977) to be equivalent to the PL model, introduced independently by Luce (1959) and Plackett (1975). When the model parameters are known, the assortment optimization problem has been shown to be efficiently solvable for the MNL model (Talluri and Van Ryzin 2004), for variants of the nested logit (NL) model (Davis et al. 2014, Gallego and Topaloglu 2014), and for the Markov chain model (Blanchet et al. 2016, Feldman and Topaloglu 2017). Due to existing hardness results for most other choice models (Bront et al. 2009, Aouad et al. 2018), Jagabathula (2014) showed in a empirical study that a local search algorithm performs very well even for instances where efficient algorithms are not known to exist. As mentioned earlier, the literature in operations management is mostly restricted to models whose choice probabilities are known to be efficiently computable. With this regard, our key
contribution is to extend a popular model in the machine learning literature to choice contexts and to assortment optimization settings.

2. Model and Problem Statement

2.1. Definitions

Notation. We consider a universe \( \mathcal{U} \) of \( n \) products. In order to distinguish these products from their corresponding ranks, we let \( \mathcal{U} = \{a_1, \ldots, a_n\} \) denote the universe of products, under an arbitrary indexing. Preferences over this universe are captured by an anti-reflexive, anti-symmetric, and transitive relation \( \succ \), which induces a total ordering (or ranking) over all products; specifically, \( a \succ b \) means that \( a \) is preferred to \( b \). We represent preferences through rankings or permutations. A complete ranking (or simply a ranking) is a bijection \( \sigma : \mathcal{U} \to [n] \) that maps each product \( a \in \mathcal{U} \) to its rank \( \sigma(a) \in [n] \), where \( [n] = \{1, \ldots, n\} \). Lower ranks indicate higher preference so that \( \sigma(a) < \sigma(b) \) if and only if \( a \succ b \), where \( \succ \) denotes the preference relation induced by the ranking \( \sigma \). For simplicity of notation, we also let \( \sigma_i \) denote the product ranked at position \( i \). Thus, \( \sigma_1 \sigma_2 \cdots \sigma_n \) is the list of the products written by increasing order of their ranks.

Mallows model. The Mallows model is a member of the distance-based ranking family models (see Murphy and Martin (2003)). This model is described by a modal ranking \( \omega \), which denotes the central or modal permutation, and a concentration parameter \( \theta \in \mathbb{R}_+ \), such that the probability of each permutation \( \sigma \) is given by

\[
\lambda(\sigma) = \frac{e^{-\theta \cdot d(\sigma, \omega)}}{\psi(\theta)}.
\]

Here, \( \psi(\theta) = \sum_\sigma \exp(-\theta \cdot d(\sigma, \omega)) \) is the normalization constant, and \( d(\cdot, \cdot) \) is the Kendall-Tau metric of distance between permutations defined as

\[
d(\sigma, \omega) = \sum_{i<j} 1[(\sigma(a_i) - \sigma(a_j)) \cdot (\omega(a_i) - \omega(a_j)) < 0].
\]

In other words, \( d(\sigma, \omega) \) counts the number of pairwise disagreements between the permutations \( \sigma \) and \( \omega \). It is easy to verify that \( d(\cdot, \cdot) \) is a distance function that is right-invariant under the composition of the symmetric group, i.e., \( d(\pi_1, \pi_2) = d(\pi_1 \pi, \pi_2 \pi) \) for every \( \pi, \pi_1, \pi_2 \), where the composition \( \sigma \pi \) is defined as \( \sigma \pi(a) = \sigma(\pi(a)) \). This symmetry can be exploited to show that the normalization constant \( \psi(\theta) \) has a closed-form expression (Marden 1995) given by

\[
\psi(\theta) = \prod_{i=1}^{n+1} \frac{1 - e^{-i\theta}}{1 - e^{-\theta}}.
\]

Note that \( \psi(\theta) \) depends on the concentration parameter \( \theta \) and on the number of products \( n \) but does not depend on the modal ranking \( \omega \).
Intuitively, the Mallows model defines a set of consumers whose preferences are “similar”, in the sense of being centered around a common permutation, where the probability for deviations thereof is decreasing exponentially. The similarity of consumer preferences is captured by the Kendall-Tau distance metric.

**Mixture of Mallows model.** More generally, the mixture of $K$ Mallows models is given by $K$ segments where, for each segment $k = 1, \ldots, K$, we are given its probability $p_k$ as well as a Mallows distribution with modal ranking $\omega_k$ and concentration parameter $\theta_k$. Therefore, the probability of any permutation $\sigma$ in the mixture model is given by

$$
\lambda(\sigma) = \sum_{k=1}^{K} p_k \cdot e^{-\theta_k \cdot d(\sigma, \omega_k)} / \psi(\theta_k).
$$

2.2. Problem statement

**Choice probability computation.** We first focus on efficiently computing the probability that a product $a$ will be chosen from an offer set $S \subseteq \mathcal{Y}$ under a given mixture of Mallows model. When offered an assortment $S$, the customer first samples a preference list according to the mixture of Mallows model and then chooses the most preferred product from $S$ according to this list. Therefore, the probability of choosing product $a$ from the offer set $S$ is given by

$$
\mathbb{P}(a|S) = \sum_{\sigma} \lambda(\sigma) \cdot 1_{[\sigma, a, S]},
$$

where $1_{[\sigma, a, S]}$ indicates whether $\sigma(a) < \sigma(a')$ for all $a' \in S, a' \neq a$. Note that the above summation runs over all $n!$ preference lists, meaning that it is a-priori unclear if $\mathbb{P}(a|S)$ can be computed efficiently.

**Assortment optimization.** Once we are able to compute the choice probabilities, we turn our attention to the assortment optimization problem under the mixture of Mallows model. In this problem, each product $a$ has an exogenously fixed price $r_a$. Moreover, there is an auxiliary product $a_q$ that represents the outside option (no-purchase), with price $r_q = 0$, which is always included in the assortment. The goal in the assortment optimization problem is to determine a feasible subset of products that maximizes the expected revenue:

$$
\max_S \mathcal{R}(S) = \max_S \sum_{a \in S} \mathbb{P}(a|S \cup \{a_q\}) \cdot r_a.
$$

As mentioned in Section 1.1, this problem is NP-hard to approximate within factor $\Omega(1/n^{1-\epsilon})$. Before tackling these computational challenges, we showcase the benefits of using a mixture of Mallows model through a case study.
3. Value of Smoothing: Case Study

In this section, we showcase the benefits of Mallows-based smoothing through a case study on real-world preference data on different sushi types. Our results show that smoothing significantly improves the accuracy of out-of-sample choice probability prediction and also of the assortment decision. They also highlight two benefits of smoothing: (a) a sparser description of the underlying choice behavior and (b) better generalization beyond the observed choices.

Sparse choice models describe the underlying choice behavior using $T$ preference lists. Our study shows that smoothing can capture the same choice behavior with far fewer than $T$ Mallows mixture components. A sparse model with $T$ preference lists can result in a nonzero choice probability for at most $T$ different products. Therefore, to capture nonzero choice probabilities under offer sets of size $\Omega(n)$ we need $T$ to scale like $\Omega(n)$. The scaling can become $\Omega(\text{poly}(n))$ depending on the number of offer sets and the complexity of the choice patterns. On the other hand, Mallows smoothing can potentially result in model with only $K$ mixture components with $K \ll T$. For instance, even for a single-class Mallows model with $K = 1$, we need $T = n$ preference lists to capture the choice probabilities of the complete assortment $[n]$. How $K$ scales with respect to $T$ is ultimately an empirical question. In our study with $n = 10$, we observe that a Mallows mixture model with $K \leq 15$ is better than a sparse choice model with $T = 1,250$ in capturing the underlying choice behavior. Of course, the degree of compression from $T$ to $K$ components can vary from application to application.

Smoothing also results in better generalization when the observed data lack sufficient variation, which is a common occurrence in practice. It is often the case that offer set variation comes from stock outs. When a retailer offering $n$ products to its customers stocks out of at most $\ell$ products at a time, we observe choices for offer sets of sizes at most $n - \ell$ products. Retailer replenishment policies often result in only a few products being stocked out at a time; therefore, we see that $\ell \ll n$. In these cases, the sparse model cannot generalize choice probability predictions to offer sets of size less than $n - \ell$. To see this, consider the case with no stock outs, so that $\ell = 0$. For a sparse model, this data can only identify the top-ranked product in each preference list. As a result, we don’t know how a customer with particular preference list might react if her most preferred product is stocked out. Smoothing, on the other hand, allows us to predict choice probabilities for all offer set sizes, but with the additional assumption of a Mallows kernel. As the variation in the data increases, the impact of smoothing becomes less prominent, which is desirable. In our study, we observe that smoothing allows us to generalize choice probability predictions to smaller offer set sizes using observations from larger offer sets.

Finally, we note that implementing the case study requires the methods we propose in the later sections to compute choice probabilities under the Mallows model and then determine the optimal
assortment decision. We describe the case study first to showcase the practical value of Mallows-based
smoothing. We provide appropriate pointers to the methods in this section, but describe them in later
sections.

Section outline. This section is organized as follows. Section 3.1 provides a detailed description of
our data set. Section 3.2 explains the estimation procedures we employ to fit the sparse distribution
over rankings and the mixture of Mallows models. Section 3.3 compares the accuracy of predicting
pairwise choice probabilities along with a wide array of additional metrics. We find that smoothing
results in significant improvements, and specifically, up to 15% when using the KL divergence metric.
Next, we show in Section 3.4 that the mixture of Mallows model extracts up to 2% more revenue than
the sparse distribution over rankings, i.e., that improved prediction accuracy translates into better
assortments decisions. Finally, in Section 3.5, we show that a mixture of Mallows model strongly
outperforms (both on prediction and decision accuracy) the mixture of MNL model, which is a very
popular choice model.

3.1. Data description

We use a publicly available dataset consisting of preference lists over different sushi types (Kamishima
et al. 2005). This data set consists of 5,000 complete rankings over 10 varieties of sushi, listed in
Table 1. Each ranking corresponds to the preferences of a survey respondent who was asked to
rank the different types of sushi according to his/her preferences. The 10 types of sushi were chosen
among the most popular ones so that respondents could produce a strict ordering of the options (see
Kamishima (2003) for additional details). This is a particularly unique data set in that it contains
the true underlying complete preference orders of the customers.

<table>
<thead>
<tr>
<th>fatty tuna</th>
<th>sea urchin</th>
<th>sea eel</th>
<th>shrimp</th>
<th>salmon roe</th>
</tr>
</thead>
<tbody>
<tr>
<td>tuna</td>
<td>squid</td>
<td>egg</td>
<td>tuna roll</td>
<td>cucumber roll</td>
</tr>
</tbody>
</table>

Table 1 List of 10 different sushis that are part of the sushi data set (Kamishima et al. 2005).

For the purposes of our study, we transform the data as follows. First, to use these data for
assortment optimization, we need to introduce a no-purchase option. In order to ensure a reasonable
market share for the no-purchase option, we add this alternative in one of the first 6 positions of each
ranking uniformly at random. To test the improvements in prediction accuracy, we randomly split
the rankings into a training set $\mathcal{T}$, consisting of 1,250 rankings, and a validation set $\mathcal{V}$, consisting of
3,750 rankings.
Second, for the purposes of training our models, we convert the rank data into choice data as follows. For a collection \( C \) of subsets, we compute the empirical choice probabilities for each subset using the training rankings, i.e., for each \( S \in C \) and \( i \in S \), we set the empirical choice probability equal to

\[
\hat{\pi}(i, S) = \frac{1}{|T|} \sum_{\sigma \in T} 1[i >_{\sigma} j : \forall j \in S \setminus \{i\}].
\]

For our study, we use a collection \( C \) of subsets with the following particular form: For \( \ell \in \{0, 1, 2, 3\} \), we consider the collection \( C_\ell \), that consists of all subsets of size at least \( n - \ell \), corresponding to settings in which at most \( \ell \) products stock out. These subsets resemble the types of offer sets commonly observed in the retail settings, in which firms maintain high service levels and, therefore, the probabilities of stock outs are low. Jagabathula and Rusmevichientong (2017) use a similar collection of offer sets in their numerical study.

### 3.2. Estimation

**Sparse distribution.** We compare the prediction accuracies of a sparse rank-based choice model and its corresponding Mallows-smoothed model. We fit a sparse distribution over rankings to the choice data generated above using the procedure developed by Jagabathula and Rusmevichientong (2017). This procedure aims to find the distribution \( \lambda \) that maximizes the likelihood of the observed data. It formulates the likelihood problem as a high-dimensional constrained convex program and solves it using the Frank-Wolfe algorithm (Frank and Wolfe 1956). The output of this procedure is a distribution \( \lambda \) over partial rankings. When the choice data are comprised of the collection \( C_\ell \) of subsets, the partial rankings output by the procedure only specify the top-\( \ell \) products. We convert a distribution over partial rankings into a sparse distribution over complete rankings by sampling a top-\( \ell \) ranking according to \( \lambda \) and then uniformly sampling the positions of the remaining \( n - \ell \) products. Since \(|T| = 1,250\), we sample 1,250 rankings as described to be the support of our sparse distribution over rankings.

**Mixture of Mallows.** Once we obtain the sparse distribution over rankings, we smooth it to obtain a mixture of Mallows models. To this end, we make use of the sparse distribution and the number \( K \) of mixture components as inputs to an expectation-maximization (EM) algorithm, which outputs a mixture of \( K \) Mallows distribution. We describe the implemented EM algorithm in Appendix A. Since \( K \) is a priori unknown, we vary \( K \) over the set \( \{4, 6, 8, 10, 15\} \); a similar range of values was used by Lu and Boutilier (2014).

### 3.3. Prediction accuracy

We evaluate the effect of smoothing on out-of-sample prediction accuracy, focusing on the task of predicting pairwise comparisons. For every pair \( i \neq j \), let \( \hat{\pi}_{sp}(i, \{i, j\}) \) and \( \hat{\pi}_{sm}(i, \{i, j\}) \) denote the
choice probabilities computed using the sparse distribution over rankings and the mixture of Mallows model, respectively. Note that in order to compute \( \hat{\pi}_{sm}(i, \{i, j\}) \), we use the procedure presented in Section 4. The ground truth choice probabilities are denoted by \( \pi_{tr}(i, \{i, j\}) \) and computed using the rankings of the validation set \( V \).

To evaluate the performance of each model, we use three popular metrics: the log-likelihood, the KL divergence, and the MAPE which are defined for \( m \in \{sp, sm\} \) as:

\[
\mathcal{L}(\hat{\pi}_m) = \sum_{i \neq j} \left[ N_{tr}(i, \{i, j\}) \cdot \log(\hat{\pi}_m(i, \{i, j\})) + N_{tr}(j, \{i, j\}) \cdot \log(\hat{\pi}_m(j, \{i, j\})) \right],
\]

\[
\mathcal{KL}(\hat{\pi}_m) = \sum_{i \neq j} \left[ \pi_{tr}(i, \{i, j\}) \cdot \log(\frac{\pi_{tr}(i, \{i, j\})}{\hat{\pi}_m(i, \{i, j\})}) + \pi_{tr}(j, \{i, j\}) \cdot \log(\frac{\pi_{tr}(j, \{i, j\})}{\hat{\pi}_m(j, \{i, j\})}) \right],
\]

\[
\text{MAPE}(\hat{\pi}_m) = \sum_{i \neq j} \left[ \frac{|\pi_{tr}(i, \{i, j\}) - \hat{\pi}_m(i, \{i, j\})|}{\pi_{tr}(i, \{i, j\})} \right],
\]

where \( N_{tr}(i, \{i, j\}) \) counts, in the validation set \( V \), the number of times that \( i \) was preferred to \( j \) when \( \{i, j\} \) was offered. Note that with this definition, the KL divergence is always non-negative and is minimized by \( \pi_{tr} \), in which case \( \mathcal{KL}(\pi_{tr}) = 0 \). For both the KL divergence and the MAPE, lower values generally indicate better predictive accuracies. For the log-likelihood, higher values indicates better predictive accuracy.

We run our experiments for different collections \( C_\ell \) and different number \( K \) of Mallows segments by varying \((\ell, K)\) over the set \( \{0, 1, 2, 3\} \times \{4, 6, 8, 10, 15\} \). For each \((\ell, K)\), we report in Table 2, 3, and 4 the percentage improvement obtained in prediction accuracy due to Mallows-based smoothing. The reported improvements are obtained using 4-fold cross validation, in which we split the data into four folds, train on three folds and then test on the remaining fold and then repeat this procedure four times, each time testing on a different fold. The results are then averaged across the four splits.

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>Number ( K ) of Mallows segments</th>
<th>Best improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>-0.06%</td>
<td>0.05%</td>
</tr>
<tr>
<td>1</td>
<td>0.68%</td>
<td>0.69%</td>
</tr>
<tr>
<td>2</td>
<td>0.36%</td>
<td>0.56%</td>
</tr>
<tr>
<td>3</td>
<td>0.26%</td>
<td>0.60%</td>
</tr>
</tbody>
</table>

Table 2 Percentage improvements in log-likelihood from smoothing \( (\mathcal{L}(\pi_{sm}) - \mathcal{L}(\hat{\pi}_{sp})/\mathcal{L}(\pi_{sp})) \).

The improvement in predictive accuracy is significant. For different values of \( K \), the improvement in KL divergence of smoothing the distribution over rankings ranges from 1.61% to 15.16%, while the improvement in MAPE ranges from 0.92% to 9.44%. The improvements in log-likelihood are positive, but smaller in magnitude (less than 1%). It is worth mentioning that these metrics only measure the improvement in prediction accuracy, without accounting for the model complexity. For this reason,
we also compute the Akaike Information Criterion (AIC), which penalises the log-likelihood using the number of parameters. The results are reported in Table EC.1 (in Appendix C). We observe that smoothing significantly improves the AIC which confirms that smoothing reduces overfitting.

It is surprising to discover that, even with a small number of Mallows segments $K$, we observe noticeable improvements in prediction accuracy. Interpreting $K$ as the number of customer types in the population, this phenomenon implies that smoothing can lead to more parsimonious models that rely only on a few customer types. It is important to note that smoothing results in an improvement for nearly all values of $K$ we examined, indicating robustness of performance to the number of customer types. In practice, one would typically tune the value of $K$ using cross validation. Here, however, we observe that any smoothing actually helps. Using a sparse distribution over rankings typically overfits the data, especially since the data are providing only partial preference information. Smoothing allows us to reduce overfitting and improves the predictive accuracy.

We conclude this section by reporting the empirical values the concentration parameter $\theta$ obtained. Table 5 reports the average value of this parameter for each $(\ell, K)$ over the set $\{0, 1, 2, 3\} \times \{4, 6, 8, 10, 15\}$, where the average is taken over the mixture components. As the number of segments increases, the value of $\theta$ decreases as the population heterogeneity is captured with more concentrated modes. We next test whether this improved accuracy in prediction leads to better decision-making, in the form of higher revenue assortments.

### 3.4. Decision accuracy

In what follows, we evaluate the improvements in decision accuracy achieved due to smoothing. For that, we determine the optimal assortments prescribed by the sparse model $\hat{\pi}_{sp}$ and by the smoothed model $\hat{\pi}_{sm}$, whose expected revenues are compared under the ground-truth model. In order to compute

#### Table 3

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Number $K$ of Mallows segments</th>
<th>Best improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>-0.54%</td>
<td>0.51%</td>
</tr>
<tr>
<td>1</td>
<td>7.91%</td>
<td>7.84%</td>
</tr>
<tr>
<td>2</td>
<td>6.19%</td>
<td>8.83%</td>
</tr>
<tr>
<td>3</td>
<td>4.02%</td>
<td>9.66%</td>
</tr>
</tbody>
</table>

#### Table 4

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Number $K$ of Mallows segments</th>
<th>Best improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>-0.23%</td>
<td>0.29%</td>
</tr>
<tr>
<td>1</td>
<td>4.23%</td>
<td>4.25%</td>
</tr>
<tr>
<td>2</td>
<td>3.68%</td>
<td>4.92%</td>
</tr>
<tr>
<td>3</td>
<td>2.14%</td>
<td>6.94%</td>
</tr>
</tbody>
</table>

we also report the empirical values the concentration parameter $\theta$ obtained. Table 5 reports the average value of this parameter for each $(\ell, K)$ over the set $\{0, 1, 2, 3\} \times \{4, 6, 8, 10, 15\}$, where the average is taken over the mixture components. As the number of segments increases, the value of $\theta$ decreases as the population heterogeneity is captured with more concentrated modes. We next test whether this improved accuracy in prediction leads to better decision-making, in the form of higher revenue assortments.
optimal assortments, we are required to set a price for each of the sushi types. Since the data set provides the average selling prices, we set the price of each type to be the average selling price, normalized such that the maximum price over all types is 1.

To solve the resulting assortment optimization problem under the sparse model $\hat{\pi}_{sp}$, we use a straightforward MILP formulation (see, for instance, Bertsimas and Mišic (2018)); for completeness, we present this MILP in Appendix B. To solve the analogous problem under the smoothed model $\hat{\pi}_{sm}$, we use the MIP presented in Theorem 2 (see Section 5). Let $S_{sp}$ and $S_{sm}$ denote the optimal assortments under the sparse and smoothed models, respectively. We compare the revenue generated by these solutions under the ground truth model $\pi_{tr}$ and report the percentage improvement in the revenues from smoothing, given by:

$$\frac{\sum_{i \in S_{sm}} r_i \cdot \pi_{tr}(i, S) - \sum_{i \in S_{sp}} r_i \cdot \pi_{tr}(i, S)}{\sum_{i \in S_{sp}} r_i \cdot \pi_{tr}(i, S)}.$$

Table 6 reports the percentage improvements for each value of $\ell \in \{0, 1, 2, 3\}$. The reported values are obtained using 4-fold cross validation and correspond to the best improvement as the number $K$ of Mallows segments is varied over $\{4, 6, 8, 10, 15\}$. We observe that smoothing not only improves the prediction accuracy, but also the decision accuracy. The best improvements are uniformly positive as $\ell$ is varied over the set $\{0, 1, 2, 3\}$. These improvements in revenues can result in substantial improvements in profits, especially in retail settings where operating margins are slim (Méndez-Díaz et al. 2010).

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Number $K$ of Mallows segments</th>
<th>Best improvement</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>0.02%</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
<td>0.00%</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>1.36%</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1.25%</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

Table 6 Percentage improvements in revenue from smoothing.
3.5. Comparison with mixture of MNL model

We conclude this numerical section by showing how a mixture of MNL model, a very popular and general choice model, performs on our dataset. Such a model is given by a distribution over $K$ different MNL models. For all $k \in [K]$ and $j \in [n]$, let $v_{j,k}$ denote the MNL preference weight of product $j$ in segment $k$, and let $p_k$ denote the probability of segment $k$. For any assortment $S \subseteq [n]$ and $j \in S \cup \{0\}$, the choice probability of product $j$ is given by

$$
\pi(j, S) = \frac{\sum_{k=1}^{K} p_k \cdot \frac{v_{j,k}}{\sum_{i \in S \cup \{0\}} v_{i,k}}}{.}
$$

To allow for meaningful comparisons, we fit mixtures of $K$ models where $K \in \{4, 6, 8, 10, 15\}$. This way, the number of parameters matches that of our mixture of Mallows. To fit the model, we employ an EM procedure as well (see Talluri and van Ryzin (2004), for instance), where we restrict the number of iterations and the precision accuracy to be similar to our EM procedure for the sparse model over rankings. Overall, we find that on this dataset, the mixture of Mallows model outperforms the mixture of MNL model approach. We report the improvements over the mixture of Mallows model from both the sparse distribution over rankings and mixture of Mallows. Consistent with existing literature (Farias et al. 2013), we find that the mixture of MNL is already outperformed by the sparse ranking model in many cases. Smoothing results in further improvements. As an illustration, the improvement in log-likelihood and revenue are reported in Tables 7 and 8; improvements on additional metrics are reported in Appendix D. We use the subscript "mnl" to denote the metrics computed using the mixture of MNL model.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{sp}$</td>
<td>1.55 %</td>
<td>1.49 %</td>
<td>2.05 %</td>
<td>2.10 %</td>
<td>2.05 %</td>
</tr>
<tr>
<td>$\Delta_{sm}$</td>
<td>6.96 %</td>
<td>7.60 %</td>
<td>7.09 %</td>
<td>7.73 %</td>
<td>2.42 %</td>
</tr>
<tr>
<td>$\Delta_{sp}$</td>
<td>6.12 %</td>
<td>6.45 %</td>
<td>3.27 %</td>
<td>3.82 %</td>
<td>7.06 %</td>
</tr>
<tr>
<td>$\Delta_{sm}$</td>
<td>8.10 %</td>
<td>8.34 %</td>
<td>4.93 %</td>
<td>5.50 %</td>
<td>6.55 %</td>
</tr>
</tbody>
</table>

Table 7 Percentage improvements in log-likelihood over mixture of MNL model

$$(\Delta_{sp} = (\mathcal{L}(\pi_{sp}) - \mathcal{L}(\hat{\pi}_{mnl}))/\mathcal{L}(\pi_{mnl}) \text{ and } \Delta_{sm} = (\mathcal{L}(\pi_{sm}) - \mathcal{L}(\hat{\pi}_{mnl}))/\mathcal{L}(\pi_{mnl})).$$

Having motivated the benefits of Mallow-based smoothing, we now turn to the fundamental challenges of computing choice probabilities and optimal assortment under this model.

4. Choice Probabilities

In this section, we show that the choice probabilities can be efficiently computed under a mixture of Mallows model, and propose two separate approaches to do so. We begin with a dynamic-programming
Table 8 Percentage improvements in revenue over mixture of MNL model from sparse distribution over rankings (Δ_{sp}) and mixture of Mallows (Δ_{sm}).

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.41 %</td>
<td>0.23 %</td>
<td>-0.90 %</td>
<td>-0.90 %</td>
<td>-0.90 %</td>
<td>-1.31 %</td>
<td>-0.67 %</td>
<td>-0.90 %</td>
<td>-2.30 %</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.18 %</td>
<td>0.18 %</td>
<td>-0.48 %</td>
<td>-1.28 %</td>
<td>-0.24 %</td>
<td>-1.04 %</td>
<td>2.54 %</td>
<td>1.66 %</td>
<td>0.08 %</td>
<td>0.82 %</td>
</tr>
<tr>
<td>2</td>
<td>-1.11 %</td>
<td>0.21 %</td>
<td>2.97 %</td>
<td>3.51 %</td>
<td>1.68 %</td>
<td>3.04 %</td>
<td>1.44 %</td>
<td>2.79 %</td>
<td>1.68 %</td>
<td>3.04 %</td>
</tr>
<tr>
<td>3</td>
<td>-1.91 %</td>
<td>-0.74 %</td>
<td>3.60 %</td>
<td>4.86 %</td>
<td>-1.75 %</td>
<td>-0.49 %</td>
<td>3.84 %</td>
<td>5.78 %</td>
<td>374.27%</td>
<td>385.70%</td>
</tr>
</tbody>
</table>

(DP) based approach. Interestingly, this approach is also instrumental to derive our MIP formulation for assortment optimization, presented in Section 5. The second approach reformulates the choice probability representation (1), which involves $n!$ summands (i.e., marginalization over all possible permutations), as a convolution of at most $n$ discrete sequences of size $n$, thereby yielding a closed-form expression of the choice probabilities. Note that without a procedure to actually compute the choice probabilities, the case study presented in Section 3 cannot be conducted.

4.1. A DP-based approach

We begin by describing our procedure for a single Mallows model and explain how it naturally extends for a mixture of Mallows model. For a single modal ranking $\omega$, we assume without loss of generality that $\omega = a_1 \cdots a_n$, i.e., the products are indexed such that the central permutation $\omega$ ranks product $a_i$ at position $i$, for all $i \in [n]$. Our approach is based on an efficient procedure to sample a random permutation according to the Mallows distribution introduced by Lu and Boutilier (2014). Algorithm 1 describes their repeated insertion method (RIM).

**Algorithm 1 Repeated Insertion Method (Lu and Boutilier 2014)**

1: Let $\sigma = \{a_i\}$.
2: For $i = 2, \ldots, n$, insert $a_i$ into $\sigma$ at position $s = 1, \ldots, i$ with probability
   \[
   \alpha_{s,i} = \frac{e^{-\theta (i-s)}}{1 + e^{-\theta} + \cdots + e^{-\theta (i-1)}}.
   \]
3: Return $\sigma$.

In each step $i$, the procedure inserts product $a_i$ in position $s$ with probability $\alpha_{s,i}$, noting that $(\alpha_{s,i})_{s=1}^i$ is an exponentially increasing sequence in $s$. Consequently, the most likely event is that product $a_i$ is placed in position $i$, i.e., in agreement with $\omega$. The procedure then allows for perturbation with exponentially decreasing probability in $i-s$, the number of disagreement with already inserted products induced by the new product. This is indeed consistent with our definition of a Mallows distribution where the probabilities of rankings decrease exponentially with the number of disagreements with $\omega$. 

Electronic copy available at: https://ssrn.com/abstract=3172997
Lemma 1 (Lu and Boutilier (2014), Thm. 3). Algorithm 1 generates a random sample from the Mallows distribution with modal ranking \( \omega \) and concentration parameter \( \theta \).

Based on the correctness of this procedure, we describe a dynamic program for computing the choice probabilities with respect to a general offer set \( S \). The key idea is to decompose these probabilities to include the position at which a product is chosen. Specifically, for \( i \leq m \leq n \) and \( s \in [m] \), let \( \pi(i, s, m) \) be the probability that product \( a_i \) is chosen (i.e., appears first among products in \( S \)) at position \( s \) at the end of step \( m \) of Algorithm 1. In other words, \( \pi(i, s, m) \) corresponds to the choice probability when restricting the universe \( \mathcal{U} \) to the first \( m \) products, \( a_1, \ldots, a_m \). With this notation, we clearly have

\[
\mathbb{P}(a_i|S) = \sum_{s=1}^{n} \pi(i, s, n).
\]

We compute the probabilities \( \pi(i, s, m) \) iteratively for \( m = 1, \ldots, n \), by relying on the sampling procedure in Algorithm 1. Starting from a permutation \( \sigma \) that consists of the products \( a_1, \ldots, a_m \), the next product \( a_{m+1} \) is inserted at position \( j \) with probability \( -\frac{j}{m+1} \). To determine the effect of this insertion on the choice probabilities \( \pi(i, s, m+1) \), \( i, s \in [m+1] \), we consider two cases, depending on whether \( a_{m+1} \) belongs to the assortment \( S \) or not.

Case 1: \( a_{m+1} \notin S \). In this case, \( \pi(m+1, s, m+1) = 0 \) for all \( s \in [m+1] \). Moreover, for any \( i \in [m] \), product \( a_i \) will be chosen at position \( s \) after \( a_{m+1} \) is inserted if and only if one of the following two mutually exclusive events occurs:

(i) \( a_i \) was chosen at position \( s \) before \( a_{m+1} \) is inserted, and \( a_{m+1} \) is inserted at position \( \ell > s \).

(ii) \( a_i \) was chosen at position \( s - 1 \), and \( a_{m+1} \) is inserted at position \( \ell \leq s - 1 \).

Consequently, we have that for all \( i \leq m \),

\[
\pi(i, s, m+1) = \sum_{\ell=s+1}^{m+1} \alpha_{\ell, m+1} \cdot \pi(i, s, m) + \sum_{\ell=1}^{s-1} \alpha_{\ell, m+1} \cdot \pi(i, s-1, m) \\
= (1 - \gamma_{s, m+1}) \cdot \pi(i, s, m) + \gamma_{s-1, m+1} \cdot \pi(i, s-1, m),
\]

where \( \gamma_{s, m} = \sum_{\ell=1}^{s} \alpha_{\ell, m} \) for all \( m, s \).

Case 2: \( a_{m+1} \in S \). In this case, for any \( i \in [m] \), product \( a_i \) is chosen at position \( s \) only if it was already chosen at position \( s \) and \( a_{m+1} \) is inserted at a position \( \ell > s \). Therefore, for all \( i \leq m \),

\[
\pi(i, s, m+1) = (1 - \gamma_{s, m+1}) \cdot \pi(i, s, m) + \gamma_{s, m+1} \cdot \pi(i, s, m+1).
\]

Further, product \( a_{m+1} \) is chosen at position \( s \) only if all products \( a_i, i \in [m] \), were chosen at positions \( \ell \geq s \) and \( a_{m+1} \) is inserted at position \( s \), implying that

\[
\pi(m+1, s, m+1) = \alpha_{s, m+1} \cdot \sum_{i=1}^{m} \sum_{\ell=s}^{n} \pi(i, \ell, m).
\]

Algorithm 2 summarizes this procedure.
Algorithm 2 DP for computing choice probabilities under the Mallows model

1: Let $S$ be a general offer set. Without loss of generality, we assume that $a_1 \in S$.

2: Let $\pi(1,1,1) = 1$.

3: For $m = 1, \ldots, n - 1$,
   
   (a) For all $i \leq m$ and $s = 1, \ldots, m + 1$, let
   $$\pi(i, s, m + 1) = (1 - \gamma_{s,m+1}) \cdot \pi(i, s, m) + \mathbb{1}[a_{m+1} \notin S] \cdot \gamma_{s-1,m+1} \cdot \pi(i, s - 1, m).$$  
   
   (b) For $s = 1, \ldots, m + 1$, let
   $$\pi(m + 1, s, m + 1) = \mathbb{1}[a_{m+1} \in S] \cdot \alpha_{s,m+1} \cdot \sum_{\ell = s}^{n} \pi(i, \ell, m).$$

4: For all $i \in [n]$, return $P(a_i | S) = \sum_{s=1}^{n} \pi(i, s, n)$.

**Theorem 1.** For any offer set $S$, Algorithm 2 returns the choice probabilities under a Mallows distribution with modal ranking $\omega$ and concentration parameter $\theta$.

Algorithm 2 has an $O(n^3)$ running time for computing $P(a_i | S)$ for all products $a \in S$ simultaneously. The dynamic program above extends to a mixture of $K$ Mallows segments in a straightforward way, by running it separately for each segment and then averaging the resulting choice probabilities, weighted by their mixture proportions $p_1, \ldots, p_K$. The resulting algorithm runs in $O(Kn^3)$ time. This is a dramatic improvement over the straightforward expression (1) that defines the choice probabilities through $n!$ summands. Moreover, as explained in Section 5, these ideas lead to an MIP formulation to solve the assortment optimization problem.

### 4.2. A closed-form expression

Before explaining how to build on the DP approach to formulate a compact MIP formulation for assortment optimization, we mention an alternative approach to compute the choice probabilities. From a running time perspective, this approach allows us to shave a factor of $n$, and compute any choice probability in $O(n^2 \log n)$ time, versus $O(n^3)$ for the DP-based algorithm. The key idea is to use certain symmetries of the Mallows distribution to reformulate the choice probabilities as a discrete convolution. A detailed discussion of this approach is presented in Appendix E.

### 5. MIP for Assortment Optimization

Building on Algorithm 2, we now present a mixed-integer programming (MIP) formulation for the assortment optimization problem. Starting with a single-segment Mallows model, we present an MIP consisting of $O(n^3)$ variables and constraints, where only $O(n)$ of the variables are binary. We then extend this formulation to a general mixture of Mallows model. Initially, we assume that the
minimum-index product $a_1$ belongs to the optimal assortment. In what follows, we focus on the MIP for the unconstrained problem, noting that our formulation can readily handle any (business) constraints that are linear in the decision variables, such as cardinality and capacity constraints, known commitments to include certain products, etc.

**Converting the DP recursion.** With the simplifying assumption above, for every $i \in [n]$ we define a corresponding decision variable $x_i \in \{0, 1\}$, with the convention that $x_i = 1$ if product $a_i$ is included in the assortment and $x_i = 0$ otherwise. Similarly to Section 4.1, let $\pi_{i,s,m}$ denote the probability that product $a_i$ will be chosen at position $s$ when the product universe is restricted to $\{a_1, \ldots, a_m\}$. Since $\mathbb{1}[a_{m+1} \in S] = x_{m+1}$ and $\mathbb{1}[a_{m+1} \notin S] = 1-x_{m+1}$ for any $m$, it follows from the correctness of Algorithm 2 that the choice probabilities $\pi_{i,s,m}$ satisfy the following nonlinear constraints:

$$
\pi_{i,s,m+1} = (1 - \gamma_{s,m+1}) \cdot \pi_{i,s,m} + (1 - x_{m+1}) \cdot \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m} \quad \forall m < n, \forall i, s \in [m]
$$

$$
\pi_{m+1,s,m+1} = x_{m+1} \cdot \alpha_{s,m+1} \cdot \sum_{i \leq m} \sum_{\ell = s}^n \pi_{i,\ell,m} \quad \forall m < n, \forall s \in [m]
$$

**Capturing multiplication of variables through auxiliary variables.** To deal with multiplications of $x$ and $\pi$ variables in these constraints, we let $y_{i,s,m+1} = (1 - x_{m+1}) \cdot \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m}$, for all $i, s \in [m]$ and $m < n$. In addition, let $z_{s,m+1} = x_{m+1} \cdot \alpha_{s,m+1} \cdot \sum_{i \leq m} \sum_{\ell = s}^n \pi_{i,\ell,m}$, for all $s \in [m]$ and $m < n$. Noting that our objective is to maximize the expected revenue $\sum_{i,s} r_i \cdot \pi_{i,s,n}$, we obtain the following non-linear integer program:

$$
\max_{x, \pi, y, z} \sum_{i,s} r_i \cdot \pi_{i,s,n}
$$

s.t. $\pi_{i,s,m+1} = (1 - \gamma_{s,m+1}) \cdot \pi_{i,s,m} + y_{i,s,m+1}$, $\forall m < n, \forall i, s \in [m]$

$0 \leq y_{i,s,m+1} \leq (1 - x_{m+1}) \cdot \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m}$, $\forall m < n, \forall i, s \in [m]$

$\pi_{m+1,s,m+1} = z_{s,m+1}$, $\forall m < n, \forall s \in [m]$

$0 \leq z_{s,m+1} \leq x_{m+1} \cdot \alpha_{s,m+1} \cdot \sum_{i \leq m} \sum_{\ell = s}^n \pi_{i,\ell,m}$, $\forall m < n, \forall s \in [m]$

$\pi_{1,1,1} = 1, \pi_{1,s,1} = 0$, $\forall s \geq 2$

$\pi_{m,s,m} = 0$, $\forall m > 1, \forall s \geq m + 1$

$\pi_{i,s,m} = 0$, $\forall m, s, \forall i > m$

$x_1 = 1, x_q = 1, x_i \in \{0, 1\}$

In this formulation, we use inequalities in the constraints corresponding to $y_{i,s,m+1}$ and $z_{s,m+1}$ since it is always optimal to set these variables at their upper bounds as all objective function coefficients are non-negative. We set $x_1 = 1$ and $x_q = 1$ since product $a_1$ is always part of the optimal solution (according to our assumption) and since the no-purchase option $a_q$ is always offered.
Linearizing the constraints. Now, since the $x$-variables are binary, the constraints corresponding to $y_{i,s,m+1}$ and $z_{s,m+1}$ can be converted into linear form by noting that

$$0 \leq y_{i,s,m+1} \leq (1 - x_{m+1}) \cdot \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m}$$

is equivalent to

$$y_{i,s,m+1} \leq \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m} \quad \text{and} \quad 0 \leq y_{i,s,m+1} \leq \gamma_{s-1,m+1} \cdot (1 - x_{m+1})$$

and similarly,

$$0 \leq z_{s,m+1} \leq x_{m+1} \cdot \alpha_{s,m+1} \cdot \sum_{s \leq m} \sum_{t=s}^{n} \pi_{i,t,m}$$

is equivalent to

$$z_{s,m+1} \leq \alpha_{s,m+1} \cdot \sum_{s \leq m} \sum_{t=s}^{n} \pi_{i,t,m} \quad \text{and} \quad 0 \leq z_{s,m+1} \leq \alpha_{s,m+1} \cdot x_{m+1}.$$

Relaxing the minimum-index product assumption. The last step consists of relaxing the assumption that $a_1 \in S^*$. To this end, we define for every product $i \in [n]$ a binary decision variable $f_i$ such that $f_i = 1$ if product $i$ is the minimum-index product of $S^*$. The following set of constraints ensures that the variables $f_i$ model the minimum-index product:

$$f_i \leq x_i, \quad \forall i \leq q$$

$$f_i \leq 1 - \sum_{j<i} f_j, \quad \forall i \leq q$$

$$\sum_{i=1}^{q} f_i = 1$$

The initial constraint $\pi_{1,1,1} = 1$ is accordingly replaced by $\pi_{i,i,i} = f_i$ for all $i \leq q$. Indeed, we know that the no-purchase option belongs to any feasible assortment. Note that one could just guess the minimum-index product in an optimal assortment; since there are $O(n)$ such guesses, this idea leads to a polynomial-time procedure. However, it does not generalize well to a mixture of Mallows model as the number of guesses grows exponentially with the number of segments $K$.

Summary. Putting everything together, we obtain the following MIP for unconstrained assortment optimization under a single-segment Mallows model:

$$\max_{x,y,z} \sum_{i,s} r_i \cdot \pi_{i,s,n}$$

s.t.

$$\pi_{i,s,m+1} = (1 - \gamma_{s,m+1}) \cdot \pi_{i,s,m} + y_{i,s,m+1}, \quad \forall m < n, \forall i, s \in [m]$$

$$y_{i,s,m+1} \leq \gamma_{s-1,m+1} \cdot \pi_{i,s-1,m}, \quad \forall m < n, \forall i, s \in [m]$$

$$0 \leq y_{i,s,m+1} \leq \gamma_{s-1,m+1} \cdot (1 - x_{m+1}), \quad \forall m < n, \forall i, s \in [m]$$
\[\begin{align*}
\pi_{m+1,s,m+1} &= z_{s,m+1}, & \forall m < n, \forall s \in [m] \\
\alpha_{s,m+1} \leq \sum_{i \leq m} \sum_{\ell = s}^{n} \pi_{i,\ell,m}, & \forall m < n, \forall s \in [m] \\
0 \leq z_{s,m+1} \leq \alpha_{s,m+1} \cdot x_{m+1}, & \forall m < n, \forall s \in [m] \\
\pi_{i,i,i} &= f_{i}, & \forall i \leq q \\
\pi_{m,s,m} &= 0, & \forall m, \forall s \geq m + 1 \\
\pi_{i,s,m} &= 0, & \forall m, s, \forall i > m \\
f_{i} &\leq x_{i}, & \forall i \leq q \\
f_{i} &\leq 1 - \sum_{j<i} f_{j}, & \forall i \leq q \\
\sum_{i \leq q} f_{i} = 1 \\
x_{q} = 1, x_{i} \in \{0, 1\}, f_{i} \in \{0, 1\}.
\end{align*}\]

**Theorem 2.** The MIP in (5) computes an optimal solution to the unconstrained assortment optimization problem under a Mallows model.

It is important to note that our MIP formulation easily extends to the mixture of Mallows model by creating a copy of each variable for each segment (for completeness, we give the formulation in Appendix F). In Section 6, by exploiting ideas of similar nature, we build on this formulation to obtain a stronger “collapsed” formulation with \(O(n^2)\) variables (rather than \(O(n^3)\)), of which only \(O(n)\) are binary. We also implement a variable bound strengthening. The combination of these improvements allows us to scale the MIP formulation to large sets of products.

**6. Performance of MIP**

In this section, we conduct a computational study to demonstrate that our MIP formulation scales well to practical problem sizes. For a broad set of ground-truth model instances, we examine the running times of this formulation to understand how it scales with core problem parameters, such as the number of products \(n\), the number of segments \(K\), and the concentration parameter \(\theta\).

**6.1. Simulation setup**

For a given configuration of the number of products \(n\) and the number of segments \(K\), we randomly generate a ground-truth model instance by sampling product prices independently and uniformly at random from the interval \([0, 1]\) and the modal rankings uniformly at random over all the \(n!\) permutations.

All the MIPs are solved using Gurobi Optimizer v. 6.0.0 on a standard desktop computer with 2.4GHz Intel Core i5, 8GB RAM, running Mac OSX El Capitan. A time limit of 1 hour is enforced for each instance when solving the MIP.
6.2. Results and discussion

We start by presenting a stronger “collapsed” formulation. We then gain insights into the inner-workings of the MIP for a single Mallows model by showing how its running time depends on two parameters: the position of the no-purchase option and the concentration parameter. We then present further results for a general mixture of Mallows model.

6.2.1. Collapsed MIP formulation. For our computational experiments, we do not directly use the formulation presented in Theorem 2, but instead, utilize a stronger collapsed formulation. We present the latter for the basic Mallows model with a single segment, noting that it can easily be extended to the mixture of Mallows model. The key idea to obtaining this formulation is to only maintain the aggregate choice probabilities for every position \( s \) and step \( m \), and to drop the dependence on the product \( i \). Instead of having variables \( \pi(i, s, m) \) for every product \( i \), we only maintain two sets of variables, \( \pi(s, m) \) and \( re(s, m) \), with the following interpretation: \( \pi(s, m) \) is the probability of a product being picked in position \( s \) after the \( m \)-th step of Algorithm 1. Also, \( re(s, m) \) is the revenue from the product picked in position \( s \) after the \( m \)-th step. We describe the full formulation in Appendix G. This collapsed formulation leads to a MIP with \( O(n^2) \) variables and constraints (instead of \( O(n^3) \) previously) and reduces the running time by an order of magnitude as illustrated in Table 9.

<table>
<thead>
<tr>
<th>( n )</th>
<th>MIP</th>
<th>collapsed MIP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average (s)</td>
<td>Max (s)</td>
</tr>
<tr>
<td>15</td>
<td>2.87</td>
<td>4.13</td>
</tr>
<tr>
<td>20</td>
<td>17.10</td>
<td>24.23</td>
</tr>
<tr>
<td>25</td>
<td>68.89</td>
<td>114.91</td>
</tr>
</tbody>
</table>

Table 9 Running time of the MIP (Theorem 2) and the collapsed MIP (Theorem EC.3) for various values of \( n \) when \( e^{-\theta} = 0.8 \). The average and maximum times are computed over 100 random ground-truth instances.

For the instances in Table 9, we assumed that the outside option is the top-ranked product in the modal ranking. This is characteristic of applications in which the retailer captures only a small fraction of the market and the outside option represents the (much larger) rest of the market. For instance, most customers visiting a website or a store leave without making a purchase (Chaffey 2019). Moreover, as shown in Section 6.2.2, having the no-purchase in the first position produces computationally-challenging instances of our problem where the MIP takes longer to terminate. In particular, we identify two parameters that considerably affect the running time: (a) the position of the no-purchase option and (b) the concentration parameter value.

6.2.2. Sensitivity of running time for single Mallows model. In what follows, we explore the sensitivity of the MIP running time with respect to two parameters in the case of a single Mallows model.
No-purchase at the top. As hinted by Assumption 1 of our PTAS (see Section 7), the position of the no-purchase option in the modal ranking influences the tractability of the underlying assortment optimization problem. We therefore wish to understand its effect on the running time of our MIP as well. For this purpose, we vary the position of the no-purchase option in the modal ranking and solve the assortment optimization problem using the MIP formulation. Note that for a single Mallows model, the binary decision variables $f_i$ introduced for Theorem 2 are redundant. Instead, we can guess the position of the minimum-index product of $S^*$. This amounts to solving several independent MIPs, which can be executed in parallel. In general, the number of MIPs that should be solved is bounded by the rank of the outside option. Since these can be run in parallel, we compare, for each possible position of the no-purchase option, the maximum running time over these many MIPs (each corresponding to a guess of the minimum-index product) and report the average value over 100 randomly generated instances.

![Figure 1](https://ssrn.com/abstract=3172997)

**Figure 1** Average value of the maximum running time of the MIPs (solid line) as a function of the position of the no-purchase option for $n = 25$ and $e^{-\theta} = 0.8$. For the same setup, a sparsity constraint with $\epsilon = 0.5$ is added (dashed line). For each position, the average is computed over 100 random ground-truth instances.

Figure 1 shows that, indeed, the average maximum running time depends on the position of the no-purchase options. In particular, the solid line in this figure shows that when the no-purchase option is either ranked the highest or the lowest, the running time of the MIP is highest. This indicates that the hardest instances for our MIP are when the outside option is ranked either first or last.
Interestingly, when the no-purchase is ranked among the last few options, we can leverage the insights gains from our PTAS to speed up the running time. Indeed, inspired by the sparsity property derived in Section 7, we can impose a constraint on the number of products which are ranked higher than the no-purchase option in the assortment. As derived in the proof of Theorem 3, imposing a constraint that the MIP can only select up to $1/\epsilon$ products preferred to the no-purchase option in the modal ranking guarantees a $(1-\epsilon)$-approximate solution. This is a straightforward linear constraint that can be added to the MIP formulation, and turns out to considerably speed up the MIP, particularly when the no-purchase is ranked toward the end of the modal ranking. The dashed line in Figure 1 illustrates the improvement when adding this sparsity constraint for $\epsilon = 0.5$. In light of these experiments, we assume for the remainder of our experiments that the no-purchase option is ranked at the top of the modal ranking; these are difficult instances, for which the sparsity constraint is not helpful.

Concentration parameter. We next explore how the running time of the MIP scales with the concentration parameter $e^{-\theta}$ under a single Mallow model. Table 10 presents the running time of the collapsed formulation (Theorem EC.3 in Appendix G) for various values of $n$ and $e^{-\theta}$. As one can observe, for $e^{-\theta} = 0.8$, the MIP can be solved efficiently even when we increase $n$. However, we observe that the running time worsens as $e^{-\theta}$ is increased.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$e^{-\theta}$</th>
<th>Average (s)</th>
<th>Max (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.5</td>
<td>0.29</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>0.63</td>
<td>0.79</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>0.94</td>
<td>1.30</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.59</td>
<td>2.31</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>9.00</td>
<td>22.24</td>
</tr>
<tr>
<td>30</td>
<td>0.5</td>
<td>1.19</td>
<td>1.83</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>2.66</td>
<td>3.67</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>3.77</td>
<td>5.28</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>8.92</td>
<td>17.27</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>212.19</td>
<td>970.01</td>
</tr>
<tr>
<td>40</td>
<td>0.5</td>
<td>2.72</td>
<td>4.15</td>
</tr>
<tr>
<td></td>
<td>0.6</td>
<td>5.58</td>
<td>7.65</td>
</tr>
<tr>
<td></td>
<td>0.7</td>
<td>10.38</td>
<td>12.47</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>24.97</td>
<td>80.56</td>
</tr>
<tr>
<td></td>
<td>0.9*</td>
<td>1,857.43</td>
<td>3,593.88</td>
</tr>
</tbody>
</table>

Table 10 Running time of the collapsed MIP for various values of $n$ and $e^{-\theta}$. The average and maximum times are computed over 100 random ground-truth instances. *For $n = 40$ and $e^{-\theta} = 0.9$, 14 instances did not terminate before the one hour time limit. The reported maximum and average are computed over the 86 (out of 100) instances that terminated before the time limit. For the other set of parameters, all instances terminated within an hour.

Strengthened collapsed formulation. In order to derive an improved formulation with better running times for higher values of $e^{-\theta}$, we present a strengthening based on the Sherali-Adams
hierarchy of level one (Sherali and Adams 2013). This strengthening idea can be described as follows. We consider two constraints corresponding to each constraint in the \( m \)-th step of the dynamic programming equations (3) and (4). In particular, we multiply each constraint in the \( m \)-th step of the dynamic program by \( x_m \) and \( 1 - x_m \) and add the two resulting constraints after linearizing the resulting bilinear terms. The details of the MIP formulation obtained from the variable bound strengthening are presented in Appendix H.

The results of this strengthening method are presented in Table 11. We observe that this strengthening helps to reduce the running time of the MIP by about 30% and increases the number of instances that terminate within the one hour time limit by 8%.

<table>
<thead>
<tr>
<th></th>
<th>Average* (s)</th>
<th>Max* (s)</th>
<th># of instances terminated (1 hour)</th>
</tr>
</thead>
<tbody>
<tr>
<td>collapsed MIP</td>
<td>1,857.43</td>
<td>3,593.88</td>
<td>86/100</td>
</tr>
<tr>
<td>collapsed MIP + strengthening</td>
<td>1,226.63</td>
<td>3,565.59</td>
<td>93/100</td>
</tr>
</tbody>
</table>

Table 11 Running time of the collapsed MIP with the successive improvements for \( n = 40 \) and \( e^{-\theta} = 0.9 \). *The reported maximum and average are computed over the instances that terminated before the time limit.

**Duality gap.** While the MIP require lengthy running times for instances with high values of \( e^{-\theta} \), we observe that the Gurobi solver finds near-optimal solutions very quickly and most of its computation time is spent in reducing the duality gap. Figure 2 shows, for a number of instances, how the revenue of the best assortment found by the MIP evolves over time. In all the instances displayed, near-optimal solutions (within 3% of the optimal revenue) are found in less than 200 seconds. On the other hand, a significant amount of time is spent in reducing the duality gap to obtain a certificate of optimality. This suggests that our current MIP formulation has a large integrality gap. While we present several approaches to strengthen this formulation, it is an interesting open problem to obtain a stronger formulation where the running time is not as dependent on \( e^{-\theta} \).

**6.2.3. Mixtures of Mallows** Despite the inefficient scaling with respect to \( e^{-\theta} \), note that in practice, fitting a mixture of Mallows model instead of a single Mallows model will better capture heterogeneity in the population. Moreover, recall from Table 5 that, as the number of mixture increases, the value of \( e^{-\theta} \) decreases as the population heterogeneity is captured with more concentrated modes. The concentration parameter determines the degree of separation between the different mixture components \( k \) and \( k' \), measured as the probability of observing the modal ranking \( \omega_{k'} \) under the distribution for mixture component \( k \), i.e., \( e^{-\theta \cdot d(\omega_k, \omega_{k'})} \). The value \( e^{-\theta} = 0.5 \) results in mixture components that are well separated. Table 12 reports the running times observed for our collapsed MIP to reach optimality when \( e^{-\theta} = 0.5 \). We observe that our collapsed MIP formulation scales well in the number of products \( n \) and can be solved efficiently for practical values of \( n \) and \( K \).

Electronic copy available at: https://ssrn.com/abstract=3172997
Figure 2  Percentage of optimal revenue and duality gap obtained by best current solution as a function of time (in seconds) while solving the MIP for several instances. Each color represents a different instance. These figures are obtained using the collapsed formulation for $n = 40$ and $e^{-\theta} = 0.9$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$K$</th>
<th>Average (s)</th>
<th>Max (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1</td>
<td>0.39</td>
<td>4.21</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>2.34</td>
<td>4.26</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>6.43</td>
<td>10.96</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>8.98</td>
<td>18.25</td>
</tr>
<tr>
<td>30</td>
<td>1</td>
<td>1.80</td>
<td>11.44</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>12.36</td>
<td>590.99</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>9.89</td>
<td>21.08</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>17.58</td>
<td>34.43</td>
</tr>
<tr>
<td>40</td>
<td>1</td>
<td>2.76</td>
<td>10.80</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>27.38</td>
<td>1,633.69</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>16.81</td>
<td>43.41</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>29.12</td>
<td>46.71</td>
</tr>
<tr>
<td>50</td>
<td>1</td>
<td>5.39</td>
<td>12.37</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>21.62</td>
<td>925.66</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>31.12</td>
<td>155.26</td>
</tr>
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<td>15</td>
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<td>100</td>
<td>1</td>
<td>24.51</td>
<td>61.60</td>
</tr>
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<td></td>
<td>5*</td>
<td>36.66</td>
<td>293.00</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>50.66</td>
<td>309.49</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>195.63</td>
<td>348.28</td>
</tr>
</tbody>
</table>

Table 12  Running time of the collapsed MIP for various values of $n$ and $K$ when $e^{-\theta} = 0.5$. The average and maximum times are computed over 100 random ground-truth instances. *For $n = 100$ and $K = 5$, one instance did not terminate before the one hour time limit, in which case the reported maximum and average are computed over the remaining instances. For the other set of parameters, all instances terminated within an hour.

To complement our MIP formulation, in the next two sections we present near-optimal algorithms for the assortment optimization problem, both accompanied by strong theoretical guarantees in terms of performance and running time, unlike the MIP formulation. Specifically, the running time of our
first approximation algorithm scales linearly in the number of segments \( K \) but requires an additional technical assumption to hold. The second approach we propose has a running time which scales exponentially in the number of segments; however, it is applicable for the mixture of Mallows model in its utmost generality. Both algorithms establish and exploit various structural properties of the Mallows distribution which are interesting on their own and could be used, for instance, to speed up our MIP formulation, as illustrated in Section 6.2.2.

7. A PTAS for Assortment Optimization

In this section, we present a polynomial-time approximation scheme (PTAS) for the assortment optimization problem under the mixture of Mallows model, with an additional technical assumption. An algorithm is said to be a PTAS if, for every problem instance and error parameter \( \epsilon > 0 \), it computes an assortment with an expected revenue of at least \( 1 - \epsilon \) times the optimal revenue, running in time that is polynomial in \( n \) and \( K \); the running time may be exponential in \( 1/\epsilon \).

It should be noted that, for an arbitrary number of segments \( K \), structural assumptions are required in order to derive any non-trivial performance guarantee. Indeed, Aouad et al. (2018) proved that assortment optimization under a sparse rank-based model with \( K \geq n \) preference lists is NP-hard to approximate within factor \( \Omega(1/n^{1-\epsilon}) \). As explained in Section 1.1, this setting is subsumed as a special case by the mixture of Mallows model. Consequently, without structural assumptions, an approximation ratio of \( \Omega(1/n^{1-\epsilon}) \) is impossible to obtain in \( \text{poly}(n, K) \)-time, unless \( P = NP \).

7.1. Structural properties of the Mallows distribution

Prior to describing our algorithm, we establish two key structural properties of the Mallows distribution, on which our analysis is based. For this purpose, consider a single Mallows model with central permutation \( \omega = a_1 \cdots a_n \). We first show that, for any pair of products \((a_i, a_j)\) with \( i < j \) (i.e., \( a_i \) is preferred to \( a_j \) in \( \omega \)), we have \( \mathbb{P}(a_i \succ_{\sigma} a_j) \geq 1/2 \), where \( \sigma \) is a random permutation, drawn from a Mallows distribution. In relation to the concentration parameter \( \theta \), it is interesting to note that, when \( \theta = 0 \), the latter distribution is uniform over all \( n! \) permutations, and therefore \( \mathbb{P}(a_i \succ_{\sigma} a_j) = 1/2 \). Moreover, when \( \theta \) tends to \( \infty \), the distribution concentrates on the mode \( \omega \), meaning that \( \mathbb{P}(a_i \succ_{\sigma} a_j) = 1 \). The next result extends these extreme cases to all values of \( \theta \).

Claim 1. For any pair of products \((a_i, a_j)\) with \( i < j \), if \( \sigma \) is drawn from a Mallows distribution, we have \( \mathbb{P}(a_i \succ_{\sigma} a_j) \geq 1/2 \).

Proof. Let \( A = \{ \sigma : a_i \succ_{\sigma} a_j \} \) and \( B = \{ \sigma : a_i \prec_{\sigma} a_j \} \). We consider the bijection \( f : A \to B \) which switches \( a_i \) and \( a_j \). More precisely, for every permutation \( \sigma \in A \),

\[
  f(\sigma)(a_k) = \begin{cases} 
    \sigma(a_j) & \text{if } k = j \\
    \sigma(a_i) & \text{if } k = i \\
    \sigma(a_k) & \text{otherwise}
  \end{cases}
\]
We show that \( d(\sigma, \omega) \leq d(f(\sigma), \omega) \) for all \( \sigma \in A \), which in turn implies the desired result. To this end, note that for any \( \sigma \in A \), we have

\[
d(f(\sigma), \omega) - d(\sigma, \omega) = 1 + \sum_{k: a_i \succ a_k \succ a_j} [\xi(f(\sigma), i, k) + \xi(f(\sigma), j, k) - \xi(\sigma, i, k) - \xi(\sigma, j, k)],
\]

where \( \xi(\sigma, i, j) = \mathbb{I}[(\sigma(a_i) - \sigma(a_j)) \cdot (\omega(a_i) - \omega(a_j)) < 0] \). Since \( a_i \succ \omega a_j \), we have three cases to consider.

Case 1: \( a_i \succ \omega a_k \succ \omega a_j \). In this case, \( \xi(f(\sigma), i, k) + \xi(f(\sigma), j, k) = 2 \) and \( \xi(\sigma, i, k) + \xi(\sigma, j, k) = 0 \).

Case 2: \( a_k \succ \omega a_i \). Here, \( \xi(f(\sigma), i, k) + \xi(f(\sigma), j, k) = 1 \) and \( \xi(\sigma, i, k) + \xi(\sigma, j, k) = 1 \).

Case 3: \( a_j \succ \omega a_k \). Here, \( \xi(f(\sigma), i, k) + \xi(f(\sigma), j, k) = 1 \) and \( \xi(\sigma, i, k) + \xi(\sigma, j, k) = 1 \).

In all cases, \( \xi(f(\sigma), i, k) + \xi(f(\sigma), j, k) - \xi(\sigma, i, k) - \xi(\sigma, j, k) \geq 0 \), which concludes the proof. \( \square \)

We proceed by extending this result to a tuple of products \( (a_{i_1}, \ldots, a_{i_m}) \). More precisely, if \( i_1 < \cdots < i_m \) (i.e., \( a_{i_1} \) is the most preferred product of \( \{a_{i_1}, \ldots, a_{i_m}\} \) in \( \omega \)), then if \( \sigma \) is a random permutation drawn from a Mallows distribution, we have \( \mathbb{P}(a_{i_1} \succ_\sigma a_{i_j}, \forall j \geq 2) \geq 1/m \). Again, note that \( \mathbb{P}(a_{i_1} \succ_\sigma a_{i_j}, \forall j \geq 2) = 1/m \) when \( \theta = 0 \), while \( \lim_{\theta \to \infty} \mathbb{P}(a_{i_1} \succ_\sigma a_{i_j}, \forall j \geq 2) = 1 \).

**Claim 2.** For any tuple of products \( (a_{i_1}, \ldots, a_{i_m}) \) such that \( i_1 < \cdots < i_m \), if \( \sigma \) is drawn from a Mallows distribution, we have for any \( k < \ell \),

\[
\mathbb{P}(a_{i_k} \succ_\sigma a_{i_j}, \forall j \neq k) \geq \mathbb{P}(a_{i_\ell} \succ_\sigma a_{i_j}, \forall j \neq \ell).
\]

In particular,

\[
\mathbb{P}(a_{i_1} \succ_\sigma a_{i_j}, \forall j \neq 1) \geq \frac{1}{m} \quad \text{and} \quad \mathbb{P}(a_{i_m} \succ_\sigma a_{i_j}, \forall j \neq m) \leq \frac{1}{m}.
\]

**Proof.** Let \( A_k = \{\sigma : a_{i_k} \succ_\sigma a_{i_j}, \forall j \neq k\} \) be the set of permutations in which \( a_{i_k} \) appears first among \( a_{i_1}, \ldots, a_{i_m} \). Similarly, let \( A_\ell = \{\sigma : a_{i_\ell} \succ_\sigma a_{i_j}, \forall j \neq \ell\} \) be the set of permutations in which \( a_{i_\ell} \) appears first among \( a_{i_1}, \ldots, a_{i_m} \). Now, consider the bijection \( f : A_\ell \to A_k \) that swaps the positions of \( a_{i_\ell} \) and \( a_k \) in each permutation. More precisely, for all \( \sigma \in A_\ell \),

\[
f(\sigma)(a_j) = \begin{cases} 
\sigma(a_k), & \text{if } j = \ell \\
\sigma(a_\ell), & \text{if } j = k \\
\sigma(a_j), & \text{otherwise}
\end{cases}.
\]

Following the arguments in the proof of Claim 1, we get \( d(\sigma, \omega) \geq d(f(\sigma), \omega) \) for all \( \sigma \in A_\ell \). This implies in turn that, for \( k < \ell \),

\[
\mathbb{P}(a_{i_k} \succ_\sigma a_{i_j}, \forall j \neq k) \geq \mathbb{P}(a_{i_\ell} \succ_\sigma a_{i_j}, \forall j \neq \ell).
\]

The second part of the claim now follows by noting that \( \sum_{k=1}^m \mathbb{P}(a_{i_k} \succ_\sigma a_{i_j}, \forall j \neq k) = 1 \) and that the minimum and maximum summands, \( \mathbb{P}(a_{i_1} \succ_\sigma a_{i_j}, \forall j \neq 1) \) and \( \mathbb{P}(a_{i_m} \succ_\sigma a_{i_j}, \forall j \neq m) \), are respectively smaller and larger than the average, \( 1/m \). \( \square \)
7.2. The PTAS

For ease of exposition, we first focus on a single Mallows model and therefore drop the index corresponding to the Mallows segment. In Section 7.3, we explain how our results extend to a mixture of Mallows model. Our algorithm is based on establishing a surprising sparsity property, proving the existence of small-sized near-optimal assortments and, crucially, on utilizing certain symmetries in the Mallows distribution over permutations.

\textit{The} $1/\epsilon$-\textit{enumeration algorithm.} Given an error parameter $\epsilon > 0$, we assume without loss of generality that $1/\epsilon$ takes an integer value. Our algorithm enumerates all feasible subsets of $\mathcal{U}$ consisting of at most $1/\epsilon$ products, and returns the best candidate assortment. In other words, our algorithm returns an assortment $\hat{S}$ defined by

$$\hat{S} = \arg \max \{ R(S) : |S| \leq 1/\epsilon \} .$$

(6)

For this purpose, we mention that the revenue $R(S)$ of a given assortment $S$ can be efficiently computed since, following Section 4, the choice probability of every product can be evaluated in polynomial time.

As mentioned earlier, an additional technical assumption is required in order to derive any non-trivial performance guarantee. Specifically, we prove that our algorithm indeed identifies $(1 - \epsilon)$-approximate assortments under the following assumption.

\textbf{Assumption 1.} The no-purchase option is ranked last in the central permutation $\omega$, i.e., $q = n$.

There is a variety of settings where Assumption 1 can naturally be motivated. In particular, consider a customer buying necessity products (such as shampoo, milk, yogurt, etc.) in a convenience store. For such products, leaving empty handed is often the least favorite option. Another context where the above assumption is very natural is when products are modeling choices on a restaurant’s menu. The vast majority of customers will choose among the available menu entries, making the no-purchase option (i.e., leaving on an empty stomach) the least favorable one.

In light of the probabilistic claims in Section 7.1, since the no-purchase option is ranked last in the central permutation $\omega$ according to Assumption 1, its choice probability is the smallest with respect to any assortment. Consequently, offering a relatively small assortment of size $O(1/\epsilon)$ is sufficient to drive down the choice probability of the no-purchase. This observation will be exploited in the next theorem to prove that we indeed compute a near-optimal assortment. As previously mentioned, the extension of this result to an arbitrary number of segments is discussed in Section 7.3.

\textbf{Theorem 3.} Under Assumption 1, the $1/\epsilon$-\textit{enumeration algorithm} is a PTAS for the assortment optimization problem under a single-segment Mallows model. The running time of this algorithm is $O(n^{O(1/\epsilon)})$. 

Electronic copy available at: https://ssrn.com/abstract=3172997
Proof. We first show that the $1/\epsilon$-enumeration algorithm indeed returns a $(1-\epsilon)$-optimal solution and then argue that its running time is polynomial in $n$ for any fixed $\epsilon$. Let $S^* \subseteq [n]$ be an optimal assortment. Clearly, if $|S^*| < 1/\epsilon$, then $S^*$ is one of the candidate assortments we examine, and therefore the algorithm returns an optimal solution. In the opposite case, where $|S^*| \geq 1/\epsilon$, let $S_\epsilon \subseteq S^*$ be the set consisting of the $1/\epsilon$ highest revenue products of $S^*$, which is among the candidate assortments constructed by our algorithm. We show that $\mathcal{R}(S_\epsilon) \geq (1-\epsilon) \cdot \mathcal{R}(S^*)$ using a sample-path analysis. For this purpose, let $\sigma$ be a fixed preference list, and let $\mathcal{R}(\sigma, S)$ be the revenue obtained from $\sigma$ when the assortment $S$ is offered, i.e.,

$$\mathcal{R}(\sigma, S) = \sum_{a \in S} 1[\sigma, a, S] \cdot r_a.$$ 

We consider two cases, depending on where the no-purchase option $a_q$ is positioned in $\sigma$ relatively to the products of $S_\epsilon$.

Case 1: $a_q \succ_\sigma a_i$ for all $i \in S_\epsilon$. That is, with respect to $\sigma$, the no-purchase option is the most preferred among the products in $S_\epsilon$. In this case, the customer chooses the no-purchase option, yielding $\mathcal{R}(\sigma, S_\epsilon) = 0$. On the other hand, we upper bound the revenue of the optimal assortment $S^*$, showing that $\mathcal{R}(\sigma, S^*) \leq 2 \cdot \mathcal{R}(S^*)$. To this end, offering a single product $a_i$ is always a feasible solution, and therefore $\mathcal{R}(S^*) \geq \mathcal{R}\{\{i^*\}\}$, where $i^*$ is the highest revenue product in $S_\epsilon$. Now, $\mathcal{R}(\{i^*\}) = \mathbb{P}(a_{i^*} \succ a_q) \cdot r_{i^*} \geq r_{i^*}/2$, where the inequality follows from Claim 1 and Assumption 1. As a result, $\mathcal{R}(S^*) \geq \mathcal{R}(\{i^*\}) \geq r_{i^*}/2$. Therefore, each product in $S^* \setminus S_\epsilon$ has a revenue of at most $r_{i^*} \leq 2 \cdot \mathcal{R}(S^*)$. Since $\mathcal{R}(\sigma, S^*) = r_a$ for some $a \in S^* \setminus \{a_q\}$, it follows that $\mathcal{R}(\sigma, S^*) \leq 2 \cdot \mathcal{R}(S^*)$.

Case 2: $a_i \succ_\sigma a_q$ and $\mathcal{R}(\sigma, S_\epsilon) = r_a$, for some $a_i \in S_\epsilon$. We show that $\mathcal{R}(\sigma, S_\epsilon) \geq \mathcal{R}(\sigma, S^*)$. Indeed, suppose that there exists a product $a_j \in S^* \setminus S_\epsilon$ such that $a_j \succ_\sigma a_i$. Since $S_\epsilon$ contains the $M$ highest revenue products of $S^*$, it must be that $r_j \leq r_i$. Therefore, $\mathcal{R}(\sigma, S_\epsilon) \geq \mathcal{R}(\sigma, S^*)$.

We now combine the two cases, in order to relate between $\mathcal{R}(S_\epsilon)$ and $\mathcal{R}(S^*)$. For case 1 to happen, note that $a_q$ has to be preferred to all the products from $S_\epsilon$. From Claim 2, this event occurs with probability at most $1/|S_\epsilon| = \epsilon$. Consequently,

$$\mathcal{R}(S^*) - \mathcal{R}(S_\epsilon) = \mathbb{P}(\text{case 1}) \cdot \mathbb{E}_\sigma [\mathcal{R}(\sigma, S^*) - \mathcal{R}(\sigma, S_\epsilon) | \text{case 1}] \leq \epsilon \cdot \mathcal{R}(S^*)$$

$$+ \mathbb{P}(\text{case 2}) \cdot \mathbb{E}_\sigma [\mathcal{R}(\sigma, S^*) - \mathcal{R}(\sigma, S_\epsilon) | \text{case 2}] \leq 2 \epsilon \cdot \mathcal{R}(S^*).$$

From a running time perspective, the number of candidate assortments is $O(n^{O(1/\epsilon)})$. By Theorem EC.2, we can compute the expected revenue $\mathcal{R}(S)$ of any assortment $S$ in $O(n^3 \log n)$ time. Therefore, the overall running time of the algorithm is $O(n^{O(1/\epsilon)})$. \qed
7.3. Extensions and further comments

Extension to constrained assortment. We would like to note that our PTAS extends to a broad range of constrained assortment problems under the Mallows model. In particular, letting \( \mathcal{S} \subseteq 2^\mathcal{U} \) be a feasibility set of assortments, the \( 1/\epsilon \)-enumeration algorithm is applicable in its current form as long as \( \mathcal{S} \) satisfies the following assumption.

Assumption 2.
- (Membership) There exists an efficient procedure that, for any \( S \subseteq \mathcal{U} \), decides whether \( S \in \mathcal{S} \) or not.
- (Downward closure) If \( S \in \mathcal{S} \) and \( T \subseteq S \), then \( T \in \mathcal{S} \).

This is a fairly general assumption, satisfied for a large class of constraints including cardinality, multi-dimensional knapsack, and matroid constraints. Note that since we enumerate over all candidate assortments of cardinality at most \( 1/\epsilon \), the only modification to be made is that of checking feasibility (i.e., membership in \( \mathcal{S} \)) for each of these assortments, which is doable by Assumption 2.

Extension to a mixture of Mallows model. Our PTAS extends to a mixture of Mallows model as long as Assumption 1 holds for each segment. Indeed, since the underlying probabilistic arguments (Claims 1 and 2) hold for each segment, the proof of Theorem 3 can easily be adapted to the case of multiple segments. Here, note that the running time scales linearly in the number of segments \( K \).

Theorem 4. Under Assumption 1, the \( 1/\epsilon \)-enumeration algorithm is a PTAS for the assortment optimization problem under a mixture of Mallows model with running time \( O(Kn^{O(1/\epsilon)}) \).

Suboptimality of sparse assortments for general instances. The PTAS presented in this section provides near-optimal performance guarantees for the mixture of Mallows model, as long as Assumption 1 holds. However, for arbitrarily-structured instances, where this assumption does not necessarily hold, sparse assortments generally cannot be near-optimal. In particular, we construct a family of instances where any \( \alpha \)-approximate assortment necessarily consists of \( \Omega(an) \) products, as stated in the next theorem. An immediate consequence of this result, whose proof appears in Appendix I, is that there are instances where the expected revenue of any \( O(1/\epsilon) \)-sized assortment is only \( O(1/n) \) times optimal.

Theorem 5. There are instances of the assortment optimization problem under a single-segment Mallows model where any \( \alpha \)-approximate assortment consists of at least \( \frac{\alpha}{128} \cdot n \) products.

We next present an approximation scheme for computing near-optimal assortments without any assumption, at the expense of greater running time.
8. A Quasi-PTAS for Assortment Optimization

In this section, we derive a quasi-polynomial-time approximation scheme (quasi-PTAS) for assortment optimization under the mixture of Mallows model. An algorithm is said to be a quasi-PTAS if, for any accuracy level $\epsilon > 0$, it computes an assortment with an expected revenue of at least $1 - \epsilon$ times the optimal revenue, running in $O(n^{\text{polylog}(n)})$ time, which could potentially be exponential in $1/\epsilon$. Specifically, letting $\Delta = \frac{r_{\max}}{r_{\min}}$ be the ratio between the maximum and minimum prices, and recalling that $K$ stands for the number of underlying segments, our main result is summarized in the next theorem. (To avoid cumbersome notation, we use $O_{\epsilon}(\cdot)$ to suppress polynomial dependencies on $1/\epsilon$, meaning that $O_{\epsilon}(f(n)) = O(\text{poly}(1/\epsilon) \cdot f(n))$.)

**Theorem 6.** For any accuracy level $\epsilon > 0$, the assortment optimization problem under a $K$-segment Mallows distribution can be approximated within factor $1 - \epsilon$ of optimal. The running time of our algorithm is $O_{\epsilon}(n^{O((1/\epsilon)^D(K), K^2 \log_D(O(K))(1+\epsilon))} n \Delta))$.

From a running time perspective, due to the exponential dependency on $K$ and $\log \Delta$, we indeed attain a quasi-PTAS for a fixed number of segments $K$, as long as $\Delta = \text{poly}(n)$. It is important to mention, however, that unlike the PTAS proposed in Section 7, our quasi-PTAS is applicable to the mixture of Mallows model in its utmost generality, without any structural assumptions. At this level of generality, exponential dependence on $K$ cannot be improved. To see this, consider the sparse rank-based model, which is a special case of the Mallows mixture model. Aouad et al. (2018) show that when the support size $K = n$ (which is necessary in most applications, as argued in Section 3), it is impossible to obtain an approximation to even within $\Omega(1/n^{1-\epsilon})$ of the optimal revenue in $\text{poly}(n, K)$-time, unless $P = \text{NP}$. Therefore, exponential dependence on $K$ is necessary in the absence of other structural assumptions.

Interestingly, even though quasi-polynomial running times have become ubiquitous in approximation schemes for a wide-range of combinatorial optimization problems (see, e.g., Bansal et al. (2006), Chan and Elbassioni (2011), Adamaszek and Wiese (2015), Mustafa et al. (2015),), Theorem 6 is the first to derive a result of this nature in the context of assortment optimization. Concurrently, a very recent quasi-PTAS of Segev (2019) for a joint assortment-inventory problem with stochastic demand differs from the technical approach presented here in all ingredients involved.

**Outline.** For ease of exposition, we present our algorithmic approach in an incremental way. Specifically, Section 8.1 is devoted to establishing a number of probabilistic claims regarding the effects of certain assortment modifying operations. Then, to gain basic intuition, we first consider the simpler case of a single segment, for which a quasi-PTAS is proposed in Section 8.2. With this background in place, Appendix J provides the additional ingredients needed to address an arbitrary number of segments.
8.1. Deletions, down-shifts, and up-mappings

In what follows, we focus our attention on two basic operations – deletions and down-shifts – where products are either deleted from a given assortment or replaced by less-preferable ones. We emphasize that the probabilistic claims regarding these operations are concerned with a single segment \((\omega, \theta)\). One particular crux of our analysis in extending these ideas to an arbitrary number of segments will be to suitably utilize deletions and down-shifts, as the notion of a less-preferable product is clearly segment-dependent.

8.1.1. Deletions. The first operation we examine is the deletion of a single product. To this end, let \(S\) be a non-empty assortment, and suppose that we delete one of its products \(d\) to obtain \(S^{-d} = S \setminus \{d\}\). The next claim states that the choice probability of any remaining product can only increase.

**Claim 3.** For any assortment \(S\) and any product \(d \in S\), we have \(\mathbb{P}(a|S^{-d}) \geq \mathbb{P}(a|S)\) for every \(a \in S^{-d}\).

**Proof.** For any product \(a \in S^{-d}\), note that if \(a \succ_{\sigma} a'\) for any \(a' \in S \setminus \{a\}\) for some permutation \(\sigma\), then clearly \(a \succ_{\sigma} a'\) for any \(a' \in S \setminus \{a, d\}\) as well. In other words, \(\mathbb{I}[\sigma, a, S] \leq \mathbb{I}[\sigma, a, S^{-d}]\) for all permutations \(\sigma\), where \(\mathbb{I}[\sigma, a, S]\) indicates whether \(\sigma(a) < \sigma(a')\) for all \(a' \in S, a' \neq a\). By representation (1) of the choice probabilities, the latter inequality directly implies that \(\mathbb{P}(a|S^{-d}) \geq \mathbb{P}(a|S)\).

\(\Box\)

8.1.2. Down-shifts. The next operation we consider is that of replacing an offered product by a currently unoffered product which is less preferable according to \(\omega\). In this case, similarly to deletions, we argue that the choice probability of any other product can only increase. To formalize this claim, for a pair of products \(d_1 \in S\) and \(d_2 \notin S\), we denote \(S^{-d_1+d_2} = S^{-d_1} \cup \{d_2\} = (S \setminus \{d_1\}) \cup \{d_2\}\).

**Claim 4.** For any assortment \(S\) and any pair of products \(d_1 \in S\) and \(d_2 \notin S\) with \(\omega(d_1) < \omega(d_2)\), we have \(\mathbb{P}(a|S^{-d_1+d_2}) \geq \mathbb{P}(a|S)\) for every \(a \in S^{-d_1}\).

**Proof.** Our proof is similar in spirit to that of Claim 1. Specifically, for some product \(a \in S^{-d_1}\), let \(A = \{\sigma : a \succ_{\sigma} a', \forall a' \in S \setminus \{a\}\}\) be the set of permutations where \(a\) is the most preferred product in \(S\), and let \(B = \{\sigma : a \succ_{\sigma} a', \forall a' \in S^{-d_1+d_2} \setminus \{a\}\}\) be those where \(a\) is the most preferred product in \(S^{-d_1+d_2}\). Now, consider the bijection \(f : A \rightarrow B\) that switches between \(d_1\) and \(d_2\). Namely, for all \(\sigma \in A\),

\[
f(\sigma)(a_k) = \begin{cases} 
\sigma(d_1) & \text{if } a_k = d_2 \\
\sigma(d_2) & \text{if } a_k = d_1 \\
\sigma(a_k) & \text{otherwise}
\end{cases}
\]

Since \(\omega(d_1) < \omega(d_2)\), we have \(d(\sigma, \omega) \leq d(f(\sigma), \omega)\) for all \(\sigma \in A\), by precisely the same reasoning as the proof of Claim 1, implying the desired result. \(\Box\)
8.1.3. Up-mappings. For a pair of assortments $T$ and $S$, with $|T| \leq |S|$, we say that an injective function $\psi : T \rightarrow S$ is an up-mapping when $\omega(\psi(a)) \leq \omega(a)$ for every product $a \in T$. In other words, every product of $T$ is mapped by $\psi$ to a product in $S$ of no greater rank. Furthermore, we say that product $f \in T$ is a fixed-point of the up-mapping $\psi$ when $\psi(f) = f$. It is easy to verify that the existence of an up-mapping is equivalent to the existence of a sequence of deletions and down-shifts that transform $S$ into $T$, where none of these operations involves fixed-points of $\psi$. Therefore, a straightforward application of Claims 3 and 4 allows us to relate the choice probability of any product offered by the assortment $T$ to that of fixed-points of $\psi$ with equal or greater rank. This relation will be important for deriving the performance guarantee of our approximation scheme.

Claim 5. Let $S$ and $T$ be a pair of assortments, and suppose there exists an up-mapping $\psi : T \rightarrow S$. Moreover, suppose that $\{a, f\} \subseteq T$, where $f$ is a fixed-point of $\psi$ satisfying $\omega(f) \geq \omega(a)$. Then, $\mathbb{P}(a|T) \geq \mathbb{P}(f|S)$.

Proof. By combining $\omega(f) \geq \omega(a)$ and Claim 2, we have $\mathbb{P}(a|T) \geq \mathbb{P}(f|T)$. In addition, since there is a sequence of deletions and down-shifts that transform $S$ into $T$ without involving fixed-points, and since $f$ is a fixed-point, we have by Claims 3 and 4 that $\mathbb{P}(f|T) \geq \mathbb{P}(f|S)$. Putting these two inequalities together gives the desired result. $\square$

8.2. Quasi-PTAS for a single segment

Prior to presenting the nuts-and-bolts of our approach, we begin with a high-level overview and some intuition. Our algorithm works by “guessing” the overall structure of an optimal assortment $S^*$ by dividing this set into buckets of products that have nearly-uniform prices and choice probabilities. The key challenge is that our selection of products in each bucket affects the choice probabilities of products in other buckets. To overcome this difficulty, we propose a structured method to pick products that are ranked between the endpoints of each bucket, ensuring that the choice probabilities in other buckets do not decrease. Our method relates between choice probabilities in the resulting assortment and those in the optimal assortment $S^*$ by an appropriately defined up-mapping, allowing us to control the cumulative revenue loss.

8.2.1. Optimal assortments: Definitions and notation. Let $S^*$ be some fixed optimal assortment with respect to the single segment $(\omega, \theta)$. For purposes of analysis, we introduce a number of definitions related to $S^*$ along with some useful notation. Specifically, we begin by partitioning the optimal assortment $S^*$ into a collection of subsets. To this end, let $d^*$ be the product in $S^*$ whose rank $\omega(d^*)$ is minimal. For any pair of integers $p \geq 0$ and $\ell \geq 0$, we define $\mathcal{I}^{p}_{\ell}$ to be the set of products $a \in S^*$ for which:

1. Choice probability $\mathbb{P}(a|S^*) \in \left[ \frac{\mathbb{P}(d^*|S^*)}{(1+\epsilon)^{p+1}}, \frac{\mathbb{P}(d^*|S^*)}{(1+\epsilon)^{\ell}} \right].$
2. Price \( r_a \in [(1 + \epsilon)^p \cdot r_{\min}, (1 + \epsilon)^{p+1} \cdot r_{\min}] \).

Note that, by Claim 2, the product \( d^* \) is the most likely one out of the assortment \( S^* \), i.e., \( \mathbb{P}(d^* | S^*) \geq \mathbb{P}(a | S^*) \) for every product \( a \in S^* \). Consequently, \( \{I_{\ell}^p\}_{p, \ell} \) is indeed a partition of \( S^* \). Figure 3a illustrates this partitioning by price and choice probability.

![Partitioning of \( S^* \) for the quasi-PTAS.](image)

Focusing on a single subset, we say that \( I_{\ell}^p \) is big when \( |I_{\ell}^p| \geq 1/\epsilon \); otherwise, this subset is said to be small. We further partition every big subset \( I_{\ell}^p \) into a collection of blocks, \( B_{\ell,1}^p, \ldots, B_{\ell,M_{\ell}^p}^p \), defined as follows. Letting \( \beta_{\ell}^p = [\epsilon \cdot |I_{\ell}^p|] \geq 1 \), the first block \( B_{\ell,1}^p \) consists of the \( \beta_{\ell}^p \) products in \( I_{\ell}^p \) whose \( \omega(\cdot) \)-value is maximal, i.e., the \( \beta_{\ell}^p \) least preferred ones. Then, out of the remaining products in \( I_{\ell}^p \), the second block \( B_{\ell,2}^p \) consists of the next \( \beta_{\ell}^p \) products with largest \( \omega(\cdot) \)-values. So on and so forth, up until the last block \( B_{\ell,M_{\ell}^p}^p \) that may consist of fewer than \( \beta_{\ell}^p \) products when \( |I_{\ell}^p| \) does not divide by \( \beta_{\ell}^p \). Within each block \( B_{\ell,m}^p \), we refer to the least preferred product as \( \text{low}[B_{\ell,m}^p] \). Figure 3b illustrates this further partitioning of a big subset \( I_{\ell}^p \). We are now ready to describe our approximation scheme.

### 8.2.2. The algorithm.

Let \( \Delta = \frac{\text{max}(r_{\min}, r_{\max})}{\text{min}(r_{\min}, r_{\max})} \) stand for the ratio between the maximal and minimal prices of any product. Furthermore, let \( \mathcal{P} = \{0, \ldots, \lceil \log(1 + \epsilon) \Delta \rceil \} \) and \( \mathcal{L} = \{0, \ldots, \lceil \log(1 + \epsilon) \frac{\Delta}{\epsilon} \rceil \} \). Our algorithm constructs an assortment \( T \) as follows:

1. Guessing subset type: For every \((\ell, p) \in \mathcal{L} \times \mathcal{P}\), we guess whether the subset \( I_{\ell}^p \) is big or small.
2. Handling small subsets: Here, we simply guess all products in \( I_{\ell}^p \) and include them in \( T \).
3. Handling big subsets: In this case, we initially guess the block size \( \beta_{\ell}^p \) as well as the number of blocks \( M_{\ell}^p \). We then guess the product \( \text{low}[B_{\ell,m}^p] \) for each of the blocks \( B_{\ell,1}^p, \ldots, B_{\ell,M_{\ell}^p}^p \). Finally, for every \( m \in [M_{\ell}^p - 2] \), let \( B_{\ell,m}^p \) be the collection of products satisfying:
   
   (a) Price \( r_a \in [(1 + \epsilon)^p \cdot r_{\min}, (1 + \epsilon)^{p+1} \cdot r_{\min}] \).
(b) Rank \( \omega(a) \in [\omega(\text{low}[\mathcal{B}^p_{\ell,m+1}]) + 1, \omega(\text{low}[\mathcal{B}^p_{\ell,m}])] \).

Then, we include in \( T \) a set of \( \beta^p_{\ell} \) products picked out of \( \mathcal{B}^p_{\ell,m} \), consisting of \( \text{low}[\mathcal{B}^p_{\ell,m}] \) and \( \beta^p_{\ell} - 1 \) arbitrarily picked additional products. Note that the last step is indeed well-defined, since \( \mathcal{B}^p_{\ell,m} \) is a known set (given our guesses), and since \( \mathcal{B}^p_{\ell,m} \subseteq \mathcal{B}^p_{\ell,m} \), meaning that the latter set has at least \(|\mathcal{B}^p_{\ell,m}| - 1 = \beta^p_{\ell} - 1\) additional products to pick from (on top of \( \text{low}[\mathcal{B}^p_{\ell,m}] \)).

### 8.2.3. Analysis

The remainder of this section is devoted to proving the next result.

**Theorem 7.** For any accuracy level \( \epsilon > 0 \), the assortment optimization problem under a single-segment Mallows distribution can be approximated within factor \( 1 - \epsilon \) of optimal. The running time of our algorithm is \( O_\epsilon(n^{O_\epsilon(\log^2(n\Delta))}) \).

We begin by discussing the running time of the algorithm and then turn to proving its performance guarantee.

**Running time.** Clearly, the number of guesses required for step 1 is \( 2^{|\mathcal{I}|+|\mathcal{P}|} \). In addition, to implement step 2, since each small subset consists of at most \( 1/\epsilon \) products, the total number of guesses over all such subsets is \( O(n^{O(|\mathcal{I}|+|\mathcal{P}|/\epsilon)}) \). Finally, to implement step 3, for every big subset we make use of \( O(n) \) guesses for the block size \( \beta^p_{\ell} \), \( O(1/\epsilon) \) guesses for the number of blocks \( M^p_{\ell} \), and \( O(n) \) guesses for the least preferred product \( \text{low}[\mathcal{B}^p_{\ell,m} \) within each block. Therefore, the total number of guesses due to big subsets is \( O(n^{O(|\mathcal{I}|+|\mathcal{P}|/\epsilon)}) \cdot (1/\epsilon)^{O(|\mathcal{I}|+|\mathcal{P}|)} = O_\epsilon(n^{O(|\mathcal{I}|+|\mathcal{P}|/\epsilon)}) \). In summary, the combined number of guesses considered by our algorithm is

\[
O_\epsilon \left( n^{O(|\mathcal{I}|+|\mathcal{P}|/\epsilon)} \right) = O_\epsilon \left( n^{O((1/\epsilon) \cdot \log \Delta \cdot \log \Delta)} \right) = O_\epsilon \left( n^{O_\epsilon(\log^2(n\Delta))} \right).
\]

**Performance guarantee.** We now turn our attention to proving that the expected revenue of the resulting assortment \( T \) is near-optimal. For this purpose, we initially establish in Lemma 2 the existence of an up-mapping from \( T \) to the optimal assortment \( S^* \) that preserves certain product-specific guesses as fixed-points. This claim allows us to subsequently argue in Lemma 3 that \( \mathcal{R}(T) = (1-O(\epsilon)) \cdot \mathcal{R}(S^*) \).

**Lemma 2.** There exists an up-mapping \( \psi : T \rightarrow S^* \) that satisfies the following properties for every \((\ell, p) \in \mathcal{I} \times \mathcal{P} : \)

1. If \( \mathcal{I}^p_{\ell} \) is small, then every product in \( \mathcal{I}^p_{\ell} \) is a fixed-point of \( \psi \).
2. If \( \mathcal{I}^p_{\ell} \) is big, then \( \text{low}[\mathcal{B}^p_{\ell,m}] \) is a fixed-point of \( \psi \), for every \( m \in [M^p_{\ell} - 2] \).

**Proof.** To better understand the upcoming construction, we advise the reader to consult Figure 4. For every small subset \( \mathcal{I}^p_{\ell} \), since step 2 guesses all products in \( \mathcal{I}^p_{\ell} \), the up-mapping \( \psi \) simply maps these products to themselves. On the other hand, when \( \mathcal{I}^p_{\ell} \) is big, for every \( m \in [M^p_{\ell} - 2] \) we guess \( \text{low}[\mathcal{B}^p_{\ell,m}] \) which can therefore also be mapped by \( \psi \) to itself. At the same time, the set of \( \beta^p_{\ell} - 1 \) additional products picked out of \( \mathcal{B}^p_{\ell,m} \) will be mapped by \( \psi \) to the \( \beta^p_{\ell} - 1 \) products in
\( B_{t,m+1}^{p} \setminus \{ \text{low}[B_{t,m+1}^{p}] \} \). By definition, the \( \omega(\cdot) \)-value of any product in \( B_{t,m}^{p} \) is strictly greater than the \( \omega(\cdot) \)-value of any product in \( B_{t,m+1}^{p} \setminus \{ \text{low}[B_{t,m+1}^{p}] \} \), implying that the function \( \psi \) is indeed an up-mapping.

Figure 4 Up-mapping for a big subset \( I_t^{sp} \).

**Lemma 3.** \( \mathcal{R}(T) \geq (1 - 8\epsilon) \cdot \mathcal{R}(S^*) \).

**Proof.** We begin by establishing an upper bound on the optimal revenue \( \mathcal{R}(S^*) \), by observing that

\[
\mathcal{R}(S^*) = \sum_{(\ell,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*) + \sum_{(\ell,p) \notin \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*)
\]

\[
\leq \sum_{(\ell,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*) + r_{\max} \cdot \frac{\epsilon}{\mathcal{N}} \cdot \mathbb{P}(d^*|S^*) \cdot \sum_{(\ell,p) \notin \mathcal{L} \times \mathcal{P}} |I_t^{sp}|
\]

\[
\leq \sum_{(\ell,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*) + r_{\min} \cdot \mathbb{P}(d^*|S^*)
\]

\[
\leq \sum_{(\ell,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*) + \epsilon \cdot \mathcal{R}(S^*)
\]

Here, to obtain inequality (7), note that every subset \( I_t^{sp} \) with \( p \notin \mathcal{P} \) is necessarily empty. Otherwise, each of its products would have a price of at least \((1 + \epsilon)^{\max \mathcal{P} + 1} \cdot r_{\min} = (1 + \epsilon)^{\log_2 \Delta + 1} \cdot r_{\min} > r_{\max} \), since \( \Delta = \frac{r_{\max}}{r_{\min}} \) which is impossible. In addition, every product \( a \in I_t^{sp} \) with \( \ell > \max \mathcal{L} \) has a choice probability of \( \mathbb{P}(a|S^*) \leq \frac{\mathbb{P}(d^*|S^*)}{(1 + \epsilon)^{\max \mathcal{L} + 1}} \leq \frac{\mathbb{P}(d^*|S^*)}{(1 + \epsilon)^{\log_2 \Delta + 1}} \leq \frac{\epsilon}{\mathcal{N}} \cdot \mathbb{P}(d^*|S^*) \). Finally, inequality (8) holds since \( \mathcal{R}(S^*) \geq r_{d^*} \cdot \mathbb{P}(d^*|S^*) \geq r_{\min} \cdot \mathbb{P}(d^*|S^*) \). By rearranging the upper bound above, we have just shown that, for \( \epsilon \in (0, 1/2] \),

\[
\mathcal{R}(S^*) \leq (1 + 2\epsilon) \cdot \sum_{(\ell,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I_t^{sp}} r_a \cdot \mathbb{P}(a|S^*)
\]
Our proof proceeds by relating the right-hand-side of this upper bound to the expected revenue of the assortment $T$. For this purpose, the latter quantity can be decomposed into the revenue contributions of our choices for small and big subsets:

$$ \mathcal{R}(T) = \sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in \mathcal{I}^p_t} r_a \cdot \mathbb{P}(a|T) + \underbrace{\sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in \mathcal{I}^p_t} \sum_{m \in [M^p_t - 2]} r_a \cdot \mathbb{P}(a|T)}_{(I)} + \underbrace{\sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{m \in [M^p_t - 2]} r_a \cdot \mathbb{P}(a|T)}_{(II)}. \quad (10) $$

To obtain a lower bound on the above expression, let $\psi : T \to S^*$ be an up-mapping that satisfies the structural properties mentioned in Lemma 2. With this ingredient in place, property 1 states that every product in a small subset $\mathcal{I}^p_t$ is a fixed-point of $\psi$. Thus, by Claim 5, it follows that $\mathbb{P}(a|T) \geq \mathbb{P}(a|S^*)$ for every such product $a$, and we have

$$(I) \geq \sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in \mathcal{I}^p_t} r_a \cdot \mathbb{P}(a|S^*).$$

In the opposite case, where $\mathcal{I}^p_t$ is a big subset, the inner summation in $(II)$ can be bounded by observing that

$$ \sum_{m \in [M^p_t - 2]} \sum_{a \in \mathcal{I}^p_t} r_a \cdot \mathbb{P}(a|T) \geq (1 + \epsilon)^p \cdot r_{\text{min}}, \quad \sum_{m \in [M^p_t - 2]} \sum_{a \in T \cap \mathcal{B}^p_{t,m}} \mathbb{P}(a|T) $$

$$ \geq (1 + \epsilon)^p \cdot r_{\text{min}} \cdot \left| T \cap \mathcal{B}^p_{t,m} \right| \cdot \mathbb{P}(\text{low}[B^p_{t,m}]|S^*) \quad (12) $$

$$ \geq (1 + \epsilon)^p \cdot r_{\text{min}} \cdot \mathbb{P}(\text{low}[B^p_{t,m}]|S^*) \cdot \left| T \cap \mathcal{B}^p_{t,m} \right| \quad (13) $$

$$ \geq (1 - 4\epsilon) \cdot (1 + \epsilon)^p \cdot r_{\text{min}} \cdot \mathbb{P}(\text{low}[B^p_{t,m}]|S^*) \cdot |I^p_t| \quad (14) $$

$$ \geq \frac{1 - 4\epsilon}{(1 + \epsilon)^2} \cdot \sum_{a \in \mathcal{I}^p_t} r_a \cdot \mathbb{P}(a|S^*) \quad (15) $$

$$ \geq (1 - 6\epsilon) \cdot \sum_{a \in \mathcal{I}^p_t} r_a \cdot \mathbb{P}(a|S^*). $$

Here, inequality (11) holds since $r_a \geq (1 + \epsilon)^p \cdot r_{\text{min}}$ for every product $a \in \mathcal{B}^p_{t,m}$. To obtain inequality (12), property 2 of Lemma 2 states that, for every $m \in [M^p_t - 2]$, the product $\text{low}[B^p_{t,m}]$ is a fixed-point of $\psi$. In turn, Claim 5 implies that $\mathbb{P}(a|T) \geq \mathbb{P}(\text{low}[B^p_{t,m}]|S^*)$ for every product $a \in T \cap \mathcal{B}^p_{t,m}$, as $\text{low}[B^p_{t,m}]$ is the product with largest $\omega(\cdot)$-value over the set $\mathcal{B}^p_{t,m}$. Inequality (13) follows from Claim 2, noting that $\omega(\text{low}[B^p_{t,m}]) \geq \cdots \geq \omega(\text{low}[B^p_{t,M^p_t - 2}])$. Inequality (14) holds since $\sum_{m \in [M^p_t - 2]} |T \cap \mathcal{B}^p_{t,m}| = (M^p_t - 2) \cdot \beta^p_t$ is precisely the total size of the blocks $\mathcal{B}^p_{t,1}, \ldots, \mathcal{B}^p_{t,M^p_t - 2}$, which is at least $|I^p_t| - 2\beta^p_t = |I^p_t| - 2(1 - 4\epsilon) \cdot |I^p_t| \geq (1 - 4\epsilon) \cdot |I^p_t|$, since $|I^p_t| \geq 1/\epsilon$. Inequality (15) holds since $r_a \leq (1 + \epsilon)^{p + 1} \cdot r_{\text{min}}$.
and $\mathbb{P}(a|S^*) \leq \frac{P(a^*|S^*)}{(1+\epsilon)^p}$ for every product $a \in \mathcal{I}^p$, and since $\mathbb{P}(\text{low}[\mathcal{B}^p_{t,i}]|S^*) \geq \frac{P(d^*|S^*)}{(1+\epsilon)^{T^*}}$. Consequently, we obtain the lower bound

$$(II) \geq (1 - 6\epsilon) \cdot \sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in \mathcal{I}_t^p} r_a \cdot \mathbb{P}(a|S^*) .$$

To conclude the proof, we plug the lower bounds on $(I)$ and $(II)$ into equation (10), and get

$$\mathcal{R}(T) \geq (1 - 6\epsilon) \cdot \sum_{(t,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in \mathcal{I}_t^p} r_a \cdot \mathbb{P}(a|S^*) \geq \frac{1 - 6\epsilon}{1 + 2\epsilon} \cdot \mathcal{R}(S^*) \geq (1 - 8\epsilon) \cdot \mathcal{R}(S^*) .$$

where the second equality follows from the upper bound in inequality (9).

As previously mentioned, we present the extension of our quasi-PTAS to a mixture of Mallows model in Appendix J.

9. Conclusions

In this paper, we consider a “smoothed” generalization of the class of sparse rank-based choice models, designed to overcome some of their key limitations. Our main building block is the Mallows model, a unimodal distribution over rankings. The key challenges in employing this approach are the exponential support size of the Mallows distribution and the absence of a closed-form expression for its choice probabilities.

We first present an efficient procedure based on dynamic programming ideas to compute the choice probabilities for any assortment under the mixture of Mallows model. Furthermore, building on this characterization, we propose a compact mixed integer program (MIP) for the constrained assortment optimization problem under a general mixture of Mallows model. In order to scale our MIP, we further present a collapsed formulation as well as several strengthening ideas. This leads to an efficient approach for solving problem instances of practical nature and scale. As a future research direction, it will be interesting to devise a formulation whose running time is independent of $e^{-\theta}$.

To complement these MIP formulations, by unraveling various structural properties of the Mallows distribution, we design two near-optimal algorithms for the assortment optimization problem. The first algorithm is a polynomial-time approximation scheme (PTAS) that requires an additional technical assumption. Our PTAS can handle a broad range of constraints on feasible assortments, including cardinality, capacity, and matroid constraints. Moreover, its running time depends linearly on the number of mixtures. The second algorithm we propose is a quasi-PTAS that applies to the assortment optimization problem in its utmost generality, without any structural assumptions. However, here
the running time depends exponentially on the number of mixtures; such dependencies cannot be avoided, in light of hardness results implied by connections to the sparse rank-based choice model. Both approaches constitute the first efficient algorithms with provably near-optimal performance guarantees for assortment optimization under the Mallows or the mixture of Mallows model in such generality.

Interestingly, the tractability of assortment optimization under a single Mallows model remains a challenging open question for future research. In fact, despite our best efforts, we have not been able to establish hardness results even for a mixture of $O(1)$ segments. As previously mentioned, an arbitrary number of segments leads to $\Omega(1/n^{1-\epsilon})$-hardness; however, reductions similar in spirit to that of Aouad et al. (2018) do not seem to be applicable for constantly-many segments. In Appendix K, we provide examples, demonstrating that optimal assortments do not seem to exhibit an intuitive structure, such as being nested by price.

References


Bartholdi, John, Craig A Tovey, Michael A Trick. 1989. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and welfare* 6(2) 157–165.


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Online Appendix

Mallows-Smoothed Distribution over Rankings Approach for Modeling Choice

Appendix A: EM Algorithm for Fitting a Mixture of Mallows Distribution

In this section, we describe the standard expectation maximization (EM) algorithm used to fit a mixture of K Mallows models to a given sample of complete rankings. Let \( \sigma_1, \ldots, \sigma_T \) denote the T samples of complete rankings obtained from fitting a sparse distribution over rankings to the choice data, as described in Section 3.2. To fit the mixture distribution, we solve the following maximum likelihood problem:

\[
\max_{p, \theta, \omega} \sum_{t=1}^{T} \log(\lambda(\sigma_t)), \quad \text{where } \lambda(\sigma) = \sum_{k=1}^{K} p_k \cdot \lambda_k(\sigma) \quad \text{and } \lambda_k(\sigma) = \frac{e^{-\theta_k \cdot d(\sigma, \omega_k)}}{\psi(\theta_k)} \text{ with } \psi(\theta) = \prod_{i=1}^{n+1} \frac{1 - e^{i \theta}}{1 - e^{-\theta}}.
\]

The above problem is a-priori challenging to solve in general since, even with the modal rankings \( \omega_1, \ldots, \omega_K \) given, the objective function is non-concave in \( p \) and \( \theta \). Therefore, we adopt the popular EM heuristic to obtain an approximation (see Marden (1995) for additional details regarding the EM algorithm).

The EM algorithm starts with an initial solution and iteratively obtains an improving solution until an appropriate stopping criterion is met. Suppose \( \{(p_k, \theta_k, \omega_k): 1 \leq k \leq K\} \) is the current solution. Then, an improving solution \( \{(\hat{p}_k, \hat{\omega}_k, \hat{\theta}_k): 1 \leq k \leq K\} \) is obtained as follows.

In the Expectation step, the algorithm computes “soft counts” \( c_{tk} \), denoting the (posterior) probability that example \( t \) was generated from mixture component \( k \). This probability is computed as

\[
c_{tk} = \frac{p_k \cdot \lambda_k(\sigma_t)}{\sum_{k=1}^{K} p_k' \cdot \lambda_k'(\sigma_t)}.
\]

Then, in the Maximization step, we first set \( \hat{\theta}_k = \frac{1}{T} \sum_{t=1}^{T} c_{tk} \cdot d(\sigma_t, \omega_k) \), and then solve the following optimization problem for each \( 1 \leq k \leq K \):

\[
\max_{\omega_k, d_k} \left\{-\theta_k \cdot \sum_{t=1}^{T} c_{tk} \cdot d(\sigma_t, \omega_k) - T \log \psi(\theta_k)\right\}.
\]

It is not difficult to verify that an optimal solution to the above problem can be obtained by sequentially solving the following two problems:

\[
d^*_k = \min_{\omega_k} \sum_{t=1}^{T} c_{tk} \cdot d(\sigma_t, \omega_k) \tag{EC.1}
\]

and

\[
\hat{\theta}_k = \arg \min_{\theta_k} \{\theta_k \cdot d^*_k + T \log \psi(\theta_k)\} \tag{EC.2}
\]
The improving solution is now set as \( (\hat{p}_k, \hat{\omega}_k, \hat{\theta}_k) \). Given the optimum value \( d_k^\ast \) of (EC.1), it is easy to verify that (EC.2) is a one-dimensional convex optimization problem, and can be solved using any suitable solver. On the other hand, (EC.1) is an NP-hard problem (Bartholdi et al. 1989), for which an optimal solution is known as the Kemeny ranking with respect to \( \sigma_1, \ldots, \sigma_T \). To obtain an optimal solution to (EC.1), we formulate the following MIP:

\[
\begin{align*}
\max \quad & \sum_{1 \leq i, j \leq n+1, i \neq j} Q_{ij} \cdot x_{ij} \\
\text{s.t.} \quad & x_{ij} + x_{ji} = 1 \quad \forall 1 \leq i, j \leq n + 1, i \neq j \\
\quad & x_{ij} + x_{jr} + x_{ri} \leq 2 \quad \forall 1 \leq i, j, r \leq n + 1, i \neq j \neq r \\
\quad & x_{ij} \in \{0, 1\} \quad \forall 1 \leq i, j \leq n + 1, i \neq j,
\end{align*}
\]

where \( Q_{ij} = \sum_{t=1}^{T} c_{t, k} \cdot \mathbb{I}[\sigma_t(i) < \sigma_t(j)] \). The above optimization problem can be shown to be equivalent to (EC.1). Here, the optimal solution \( x^\ast \) encodes the pairwise comparisons of the ranking \( \hat{\omega}_k \) such that \( x^*_{ij} = \mathbb{I}[\hat{\omega}_k(i) < \hat{\omega}_k(j)] \). Since pairwise comparisons uniquely determine the ranking, \( \hat{\omega}_k \) can be constructed from \( x^* \). In this formulation, the first constraint is imposing that either \( i \) is preferred over \( j \) or \( j \) is preferred over \( i \). The second constraint is imposing transitivity.

Once we obtain the new solution, \( \{(\hat{p}_k, \hat{\omega}_k, \hat{\theta}_k) : 1 \leq k \leq K\} \), the above process is repeated until the increase in the log-likelihood value falls below a certain threshold, or until a prespecified limit on the number of iterations is reached.

Appendix B: MIP Formulation for Assortment Optimization under a Sparse Distribution over Rankings

Following the notation introduced in Section 3, a sparse distribution over rankings is given by a collection of rankings \( \sigma_1, \ldots, \sigma_K \) and a probability \( \lambda^k \) for each ranking \( \sigma_k \). Let \( x_i \) be a binary decision variable, indicating whether product \( i \) is in the assortment or not. Let \( y_{i,k} \) be a binary decision variable, indicating whether product \( i \) is the one chosen under ranking \( \sigma_k \). We can then formulate the following mixed integer linear program (MILP) to solve the assortment optimization problem:

\[
\begin{align*}
\max \quad & \sum_{k=1}^{K} \left( \lambda^k \cdot \sum_{i=1}^{n} r_i \cdot y_{i,k} \right) \\
\sum_{i=1}^{n} & y_{i,k} = 1, \quad \forall k = 1, \ldots, K \\
y_{i,k} & \leq x_i, \quad \forall k = 1, \ldots, K, \forall i = 1, \ldots, n \\
\sum_{j: a_j < a_k a_i} & y_{j,k} \leq 1 - x_i, \quad \forall k = 1, \ldots, K, \forall i = 1, \ldots, n \\
x_q & = 1, \\
x_i & \in \{0, 1\}, \quad \forall i = 1, \ldots, n \\
y_{i,k} & \in \{0, 1\}, \quad \forall k = 1, \ldots, K, \forall i = 1, \ldots, n
\end{align*}
\]
As argued by Bertsimas and Mišic (2018, Sec. 3.2), the binary requirement for \(y_{i,k}\) can be relaxed to a non-negativity constraint, since imposing \(x_i\) to be binary forces \(y_{i,k}\) to be binary in any optimal solution as well.

**Appendix C: Akaike Information Criterion**

We also compute the Akaike Information Criterion (AIC), which penalises the log-likelihood using the number of parameters. This allows accounting for model complexity. For a given \(m \in \{sm, sp\}\), we define

\[
AIC(\hat{\pi}_m) = 2 \times N(\hat{\pi}_m) - 2 \times \mathcal{L}(\hat{\pi}_m),
\]

where \(N(\hat{\pi}_m)\) represents the number of parameters used in the model \(m\). In particular, \(N(\hat{\pi}_{sm}) = K \times 11\) (each segment is specified by nine parameters for the modal ranking, one concentration parameter and one weight) and \(N(\hat{\pi}_{sm}) = 1,250 \times 10\) (each ranking in the support is specified by nine parameters and has an associated weight). Table EC.1 reports the improvement in AIC obtained from smoothing. When accounting for model complexity, the improvements become much more significant compared to the improvement in log-likelihood by themselves. As can be seen, smoothing significantly improves the AIC which confirms that smoothing reduces overfitting.

<table>
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<tr>
<td>3</td>
<td>8.11%</td>
<td>8.40%</td>
</tr>
</tbody>
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**Table EC.1** Percentage improvements in AIC from smoothing \((AIC(\pi_{sp}) - AIC(\hat{\pi}_{sm}))/AIC(\pi_{sp})\).

**Appendix D: Comparison with Mixture of MNL Model**

In Tables EC.2 and EC.3, we report the improvement of using a mixture of MNL model over the sparse distribution over rankings model for the KL-divergence and MAPE metrics. As discussed in Section 3.5, the mixture of MNL model does not perform very well on our data set.

**Appendix E: Closed-Form Expression for the Choice Probabilities**

In this section, we explain how to obtain an efficiently computable closed-form expression for the choice probabilities. Similarly to the dynamic programming approach in Section 4.1, we derive the expression for a single Mallows model and then explain how it extends to a mixture of Mallows model.
Table EC.2  Percentage improvements in $KL$ divergence over mixture of MNL model 

$\Delta_{sp} = KL(\pi_{mnl}) - KL(\hat{\pi}_{sp})/KL(\pi_{mnl})$ and $\Delta_{sm} = KL(\pi_{mnl}) - KL(\hat{\pi}_{sm})/KL(\pi_{mnl})$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Number $K$ of MNL mixtures</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
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<td>32.49%</td>
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</tbody>
</table>

Table EC.3  Percentage improvements in MAPE divergence over a mixture of MNL model 

$\Delta_{sp} = (\text{MAPE}(\pi_{mnl}) - \text{MAPE}(\hat{\pi}_{sp}))$ and $\Delta_{sm} = (\text{MAPE}(\pi_{mnl}) - \text{MAPE}(\hat{\pi}_{sm}))$.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>Number $K$ of MNL mixtures</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
<th>$\Delta_{sp}$</th>
<th>$\Delta_{sm}$</th>
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<td>20.90%</td>
<td>26.50%</td>
<td>27.53%</td>
<td>31.60%</td>
<td>4.85%</td>
<td>8.25%</td>
<td>16.67%</td>
</tr>
</tbody>
</table>

E.1. Contiguous offer set

For simplicity, we start with the case in which the offer set is contiguous, where the choice probabilities enjoy a rather simple form. Interestingly, the latter formula is reminiscent of an MNL-based choice probability with exponentially decreasing weights. Moreover, this form serves as a building block for the choice probabilities under general offer sets that we will present next. Below, for any two integers $i \leq j$, we use $[i, j]$ to denote the set $\{i, i+1, \ldots, j\}$.

**Theorem EC.1 (Contiguous offer set).** Suppose $S = \{a_{i, j} = \{a_i, \ldots, a_j\}$ for some $1 \leq i \leq j \leq n$. Then, the probability of choosing product $a_k \in S$ under the Mallows model with modal ranking $\omega$ and concentration parameter $\theta$ is given by

$$P(a_k|S) = \frac{e^{-\theta(k-i)}}{1 + e^{-\theta} + \cdots + e^{-\theta(j-i)}}.$$  

**Proof.** It suffices to show that the choice probability drops exponentially for two consecutive products in $S$. More precisely, we show that $P(a_{k+1}|S) = e^{-\theta} \cdot P(a_k|S)$ for any $i \leq k < j$. Let $A = \{\sigma : a_k \succ \sigma a_{k+1}\}$ and $B = \{\sigma : a_{k+1} \succ \sigma a_k\}$. We consider the following bijection $f: A \rightarrow B$ which swaps $a_k$ and $a_{k+1}$. More precisely, for all $\sigma \in A$,

$$f(\sigma)(a_i) = \begin{cases} 
\sigma(a_{k+1}) & \text{if } i = k \\
\sigma(a_k) & \text{if } i = k + 1 \\
\sigma(a_i) & \text{otherwise}
\end{cases}.$$  

We show that $d(f(\sigma), \omega) = d(\sigma, \omega) + 1$ for all $\sigma \in A$, which in turn implies the desired result. Note that when swapping $a_{k+1}$ and $a_k$, we induce at least one additional pairwise disagreement since $a_k$ is
preferred to $a_{k+1}$ in both $\sigma$ and $\omega$ but the reverse is true in $f(\sigma)$. This is the only added pairwise disagreement. Indeed, since $a_k$ and $a_{k+1}$ are consecutive products in $\omega$, a product which is preferred to $a_k$ is also preferred to $a_{k+1}$, and a product which is less preferred to $a_{k+1}$ is also less preferred to $a_k$. This implies that the number of pairwise disagreements induced by $a_k$ in $\sigma$ (with products other than $a_{k+1}$) is equal to the number of pairwise disagreements induced by $a_{k+1}$ in $f(\sigma)$. Similarly, the number of pairwise disagreements induced by $a_{k+1}$ in $\sigma$ is equal to the number of pairwise disagreements induced by $a_k$ in $f(\sigma)$. This shows that when swapping $a_k$ with $a_{k+1}$, the number of pairwise disagreements goes up by exactly one and concludes the proof. \qed

E.2. General offer set

The choice probability under a general offer set has a more involved structure for which additional notation is needed. For a pair of integers $1 \leq m \leq q \leq n$, define

$$\psi(q, \theta) = \prod_{s=1}^{q} \sum_{i=0}^{s-1} e^{-\theta \cdot i} \quad \text{and} \quad \psi(q, m, \theta) = \psi(m, \theta) \cdot \psi(q - m, \theta).$$

In addition, for a collection of $M$ discrete functions $h_m : \mathbb{N} \to \mathbb{R}$, $m = 1, \ldots, M$ such that $h_m(r) = 0$ for any $r < 0$, their discrete convolution is defined as

$$(h_1 \ast \cdots \ast h_m)(r) = \sum_{r_1, \ldots, r_M \geq 0 \atop \sum_{m} r_m = r} h_1(r_1) \cdots h_M(r_M).$$

**Theorem EC.2 (General offer set).** Suppose $S = a_{[i_1,j_1]} \cup \cdots \cup a_{[i_M,j_M]}$ where $i_m \leq j_m$ for $1 \leq m \leq M$ and $j_m < i_{m+1}$ for $1 \leq m \leq M - 1$. Let $G_m = a_{[i_m,j_m+1]}$ for $1 \leq m \leq M - 1$, $G = G_1 \cup \cdots \cup G_{M-1}$, and $C = a_{[i_M,j_M]}$. Then, the probability of choosing $a_k \in a_{[i_1,j_1]}$ can be written as

$$\mathbb{P}(a_k | S) = e^{-\theta \cdot (k-i_1)} \cdot \frac{\prod_{m=1}^{M-1} \psi(|G_m|, \theta)}{\psi(|C|, \theta)} \cdot (f_0 \ast \tilde{f}_1 \ast \cdots \ast \tilde{f}_t \ast f_{t+1} \ast \cdots \ast f_{M-1})(|G|),$$

where:

- for $1 \leq m \leq M - 1$,

$$f_m(r) = \begin{cases} e^{-\theta \cdot r \cdot \frac{(j_m-i_m+1+r)}{2}} \cdot \frac{\psi(|G_m|, r, \theta)}{\psi(|G_m|, r, \theta)} & \text{if } 0 \leq r \leq |G_m|, \\ 0 & \text{if } r > |G_m| \end{cases}.$$

- for $m = M$,

$$f_0(r) = \begin{cases} \psi(|C|, |G| - r, \theta) \cdot \frac{e^{\theta \cdot (|G| - r)^2/2}}{1 + e^{-\theta \cdot (|G| - r)^2/2}} & \text{if } 0 \leq r \leq |G|, \\ 0 & \text{if } r > |G| \end{cases}.$$

- $\tilde{f}_m(r) = e^{\theta \cdot r} \cdot f_m(r)$, for $1 \leq m \leq M - 1$. 

Electronic copy available at: https://ssrn.com/abstract=3172997
Before delving into the proof of this result, we provide some intuition behind the meaning of this decomposition. The picture to keep in mind is that of the subset $S$ as a union of blocks of consecutive products (in the central permutation). Each block is separated by a set of products $G_i$. The key idea in formulating the choice probability as a convolution is to partition the set of permutations by looking at how many products from $G_i$ are preferred to $a_k$ (since $G_i \cap S = \emptyset$, products in $G_i$ are allowed to be preferred to $a_k$ even in permutations where $a_k$ is chosen among products in $S$). Then, $f_i(r_i)$ measures the effect on the choice probability of having $r_i$ products from $G_i$ preferred to $a_k$. Note that depending on whether $G_i$ is positioned before or after $a_k$ in the central permutation, we use $f_i$ or $\tilde{f}_i$.

**Proof of Theorem EC.2.** At a high level, deriving the expression for a general offer set involves breaking down the probabilistic event of choosing $a_k \in S$ into simpler events for which we can use the expression given in Theorem EC.1, and then combining these expressions using the symmetries of the Mallows distribution.

For a given vector $R = (r_0, \ldots, r_M) \in \mathbb{N}^{M+1}$ such that $r_0 + \cdots + r_M = |G|$, let $h(R)$ be the set of permutations that satisfy the following two conditions: (i) among all the products of $S$, $a_k$ is the most preferred, and (ii) for every $m \in [M]$, there are exactly $r_m$ products from $G_m$ which are preferred over $a_k$. For any $\sigma \in h(R)$ and $m \in [M]$, let $\tilde{G}_m(\sigma)$ denote the set of $r_m$ products in $G_m$ that are preferred over $a_k$. With this notation, it can be seen that there are $r_0 = |G| - \sum_{m=1}^{M} r_m$ products from $G$ that are less preferred than $a_k$. We now have,

$$P(a_k | S) = \sum_{R: r_0 + r_1 + \cdots + r_M = |G|} \sum_{\sigma \in h(R)} \lambda(\sigma).$$

Under the Mallows model and the assumption that $\omega = a_1 \cdots a_n$, we have

$$\lambda(\sigma) = \frac{e^{-\theta \sum_{i<j} 1[\sigma(a_i) > \sigma(a_j)]}}{\psi(\theta)}.$$

For each $\sigma$, we can break down the sum in the exponent as follows:

$$\sum_{i<j} 1[\sigma(a_i) > \sigma(a_j)] = C_1(\sigma) + C_2(\sigma) + C_3(\sigma),$$

where:

- $C_1(\sigma)$ is the sum over pairs of products $i < j$ such that $a_j \in \tilde{G}_m(\sigma)$ for some $m \in [M]$ and $a_i \in S$.
- $C_2(\sigma)$ is the sum over pairs of products $i < j$ such that $a_i \in \tilde{G}_m(\sigma)$ for some $m \in [M]$ and $a_j \in G_{m'}(\sigma) \setminus \tilde{G}_{m'}(\sigma)$ for some $m \neq m'$.
- $C_3(\sigma)$ is the sum over the remaining pairs of products.

Electronic copy available at: https://ssrn.com/abstract=3172997
Figure EC.1 illustrates the different types of disagreements. Here we want to compute \( P(a_4|S) \) for \( S = \{a_1, a_4, a_7, a_8\} \). We give an example of permutation \( \sigma \) where \( a_4 \) is the most preferred product in \( S \) and the different types of incurred disagreements.

Figure EC.1 illustrates the different types of disagreements. For a fixed vector \( R \), we show that \( C_1(\sigma) \) and \( C_2(\sigma) \) are independent of \( \sigma \) for all \( \sigma \in h(R) \).

**Part 1.** \( C_1(\sigma) \) counts the number of disagreements (i.e., number of pairs of products that are oppositely ranked in \( \sigma \) and \( \omega \)) between some product in \( S \) and some product in \( \tilde{G}_m(\sigma) \) for any \( m \in [M] \). Given \( m \in [M] \), a product \( a_j \in \tilde{G}_m(\sigma) \) induces a disagreement with all products \( a_i \in S \) such that \( i < j \). The number of such disagreements is equal to

\[
    j_m - i_1 + 1 - \sum_{j=1}^{m-1} |G_j|,
\]

where the last term (i.e. the sum) is equal to 0 when \( m = 1 \). Therefore, summing over all pairs, the number of disagreements is given by

\[
    C_1(\sigma) = \sum_{m=1}^{M} \sum_{a_i \in S, a_j \in \tilde{G}_m(\sigma)} 1[(\sigma(a_i) > \sigma(a_j))] = \sum_{m=1}^{M} r_m \cdot \left[ j_m - i_1 + 1 - \sum_{j=1}^{m-1} |G_j| \right].
\]

**Part 2.** \( C_2(\sigma) \) counts the number of disagreements between some product in any \( \tilde{G}_m(\sigma) \) and some product in any \( G_{m'} \setminus \tilde{G}_{m'}(\sigma) \) for \( m' \neq m \). The number of all these disagreements is equal to

\[
    C_2(\sigma) = \sum_{m > m'} \sum_{a_i \in \tilde{G}_m(\sigma), a_j \in G_{m'} \setminus \tilde{G}_{m'}(\sigma)} 1[(\sigma(a_i) > \sigma(a_j))]
\]

\[
    = \sum_{m=2}^{M} \sum_{j=1}^{m-1} |G_j| - r_j
\]

\[
    = \sum_{m=2}^{M} r_m \cdot \sum_{j=1}^{m-1} |G_j| - \sum_{m=2}^{M} r_m \cdot \sum_{j=1}^{m-1} r_j
\]

\[
    = \sum_{m=2}^{M} r_m \sum_{j=1}^{m-1} |G_j| - \frac{1}{2} |G| - r_0^2 + \frac{1}{2} \sum_{m=1}^{M} r_m^2.
\]
Consequently, for all $\sigma \in h(R)$, we can write $d(\sigma, \omega) = C_1(R) + C_2(R) + C_3(\sigma)$ and therefore,

$$\Pr(a_k|S) = \sum_{R: r_0 + r_1 + \ldots + r_M = |G|} \frac{e^{-\theta(C_1(R) + C_2(R))}}{\psi(\theta)} \cdot \sum_{\sigma \in h(R)} e^{-\theta C_3(\sigma)}.$$ 

Computing the inner sum $\sum_{\sigma \in h(R)} e^{-\theta C_3(\sigma)}$ requires a similar but more involved partitioning of the permutations as well as applying the result of Theorem EC.1 regarding the contiguous case. The details are presented in Appendix E.3. In particular, we show that for a fixed vector $R$,

$$\sum_{\sigma \in h(R)} e^{-\theta C_3(\sigma)} = \psi(|G| - r_0, \theta) \cdot \psi(|S| + r_0, \theta) \cdot \frac{e^{-\theta((k-1) - \sum_{m=1}^{l-1} r_m)}}{1 + \ldots + e^{-\theta(|S| + r_0 - 1)}} \cdot \prod_{m=1}^{M} \frac{\psi(|G_m|, \theta)}{\psi(r_m, \theta) \cdot \psi(|G|_m - r_m, \theta)}.$$ 

Putting all the pieces together yields the desired result. □

Theorem EC.2 allows us to express $\Pr(a|S)$ as a discrete convolution of various sequences. This leads to an efficient way of computing the latter probability since convolution in the time domain corresponds to product in the frequency domain. Converting a discrete sequence of size $n$ from the time domain to the frequency domain can be done in $O(n \log n)$ using the fast Fourier transform (see, for instance, Cooley and Tukey (1965)). Since there are at most $n$ such sequences, this leads to a procedure that computes $\Pr(a|S)$ in $O(n^2 \log n)$ time. For a mixture of $K$ Mallows segments, we can therefore compute the choice probability $\Pr(a|S)$ in $O(Kn^2 \log n)$ time.

Finally, we would like to note that using Fourier analysis to represent distribution over rankings has been studied before (see, for instance Huang et al. (2009) and references therein). However, the Fourier transform used in this literature is not the traditional Fourier transform but rather a specific decomposition of functions over any algebraic groups. We simply use the traditional Fourier transform defined on discrete sequences. It would be interesting to investigate whether our decomposition can be interpreted through that particular lens.

### E.3. Computing $\sum_{\sigma \in h(R)} e^{-\theta C_3(\sigma)}$

In what follows, we prove that for a fixed vector $R$,

$$\sum_{\sigma \in h(R)} e^{-\theta C_3(\sigma)} = \psi(|G| - r_0, \theta) \cdot \psi(|S| + r_0, \theta) \cdot \frac{e^{-\theta((k-1) - \sum_{m=1}^{l-1} r_m)}}{1 + \ldots + e^{-\theta(|S| + r_0 - 1)}} \cdot \prod_{m=1}^{M} \frac{\psi(|G_m|, \theta)}{\psi(r_m, \theta) \cdot \psi(|G|_m - r_m, \theta)}.$$ 

We use an approach similar to the first part of the proof. Table EC.4 shows how we further partition the set of pairs within $C_3$. Recalling that $a_k$ is the most preferred among $S$ in $\sigma$, we can define the following sums which correspond to the partitioning in Table EC.4:

- $D_1(\sigma)$ is the sum of disagreements $\mathbb{I}[(\sigma(a_i) > \sigma(a_j))]$ over pairs of products $i < j$ such that either $j = k$ and $a_k \succ a_i$, or $a_k \succ a_i$ and $a_k \succ a_j$.
- $D_2(\sigma)$ is the sum of disagreements $\mathbb{I}[(\sigma(a_i) > \sigma(a_j))]$ over pairs of products $i < j$ such that $a_i \succ a_k$ and $a_j \succ a_k$. 


For all $m \in [M]$, $D_3(\sigma, m)$ is the sum of disagreements $\mathbb{1}[(\sigma(a_i) > \sigma(a_j))]$ over pairs of products $i < j$ such that $a_j \in G_m$ and $a_i \in G_m \setminus G_m$.

We can therefore write

$$\sum_{\sigma \in h(R)} e^{-\theta \cdot C_3(\sigma)} = \sum_{\sigma \in h(R)} e^{-\theta \cdot (D_1(\sigma) + D_2(\sigma) + \sum_{m \in [M]} D_3(\sigma, m))}.$$

<table>
<thead>
<tr>
<th>$G_m$</th>
<th>$G_m'$</th>
<th>$G_m \setminus G_m$</th>
<th>$G_m' \setminus G_m'$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_m$</td>
<td>$D_2(\sigma)$</td>
<td>$D_3(\sigma)$</td>
<td>$C_2(\sigma)$</td>
<td>$C_1(\sigma)$</td>
</tr>
<tr>
<td>$G_m \setminus G_m$</td>
<td>$-$</td>
<td>$-$</td>
<td>$D_1(\sigma)$</td>
<td>$D_1(\sigma)$</td>
</tr>
<tr>
<td>$S$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$D_1(\sigma)$</td>
</tr>
</tbody>
</table>

**Table EC.4** Each table entry refers to the sum in which the disagreement between the row product and column product is counted in.

Let $\Gamma_R$ denote the set of tuples $\{ (\tilde{G}_1, \ldots, \tilde{G}_M) : \tilde{G}_m \subseteq G_m, |\tilde{G}_m| = r_m \ \forall m \in [M] \}$. For each $\gamma = (\tilde{G}_1, \ldots, \tilde{G}_M) \in \Gamma_R$, let $t(\gamma)$ denote the permutations $\sigma$ which satisfy the following two conditions:

- $\sigma \in h(R)$.
- For all $m \in [M]$, the subset of products from $G_m$ which is preferred to $a_k$ is exactly $\tilde{G}_m$.

With this notation, we can further break down the sum of interest as follows:

$$\sum_{\sigma \in h(R)} e^{-\theta \cdot C_3(\sigma)} = \sum_{\gamma \in \Gamma_R} \sum_{\sigma \in t(\gamma)} e^{-\theta \cdot (D_1(\sigma) + D_2(\sigma) + \sum_{m \in [M]} D_3(\sigma, m))}.$$

Note that, for a fixed $\gamma$, the value of $D_3(\sigma, m)$ is constant for all $\sigma \in t(\gamma)$. Indeed, each $\gamma$ induces a partition of $G_m$ such that products in $\tilde{G}_m$ are preferred to products in $G_m \setminus \tilde{G}_m$ (since $\tilde{G}_m$ are preferred to $a_k$ and $a_k$ is preferred to $G_m \setminus \tilde{G}_m$). Since $D_3(\sigma, m)$ only involves comparison of pairs where one product belongs to $G_m$ and the other to $\tilde{G}_m$, the value of $D_3(\sigma, m)$ is constant for all $\sigma \in t(\gamma)$ and only depends on $\tilde{G}_m$. Consequently, we can write

$$\sum_{\sigma \in h(R)} e^{-\theta \cdot C_3(\sigma)} = \sum_{\gamma \in \Gamma_R} e^{-\theta \cdot \sum_{m \in [M]} D_3(\tilde{G}_m)} \sum_{\sigma \in t(\gamma)} e^{-\theta \cdot (D_1(\sigma) + D_2(\sigma))}.$$

Note that the two terms $D_1(\sigma)$ and $D_2(\sigma)$ involve separate subset of products. Consequently, the ordering of products in $\cup_m \tilde{G}_m$ does not affect $D_2(\sigma)$ and similarly, the ordering of products in $S \cup (\cup_m (G_m \setminus \tilde{G}_m))$ does not affect $D_1(\sigma)$. This implies that

$$\sum_{\sigma \in t(\gamma)} e^{-\theta \cdot (D_1(\sigma) + D_2(\sigma))} = \left( \sum_{\sigma \in A} e^{-\theta \cdot D_1(\sigma)} \right) \cdot \left( \sum_{\sigma \in B} e^{-\theta \cdot D_2(\sigma)} \right),$$

where $A$ is the set of permutations over products in $\cup_m \tilde{G}_m$ and $B$ is the set of permutations over products in $S \cup (\cup_m (G_m \setminus \tilde{G}_m))$ where $a_k$ is the most preferred product. Note that the first term corresponds to the normalization constant of a Mallows model over only products in $\cup_m \tilde{G}_m$ whereas...
the second term is the numerator of the choice probability of choosing \( a_k \) among only products in 
\( S \cup (\cup_m (G_m \setminus \widehat{G}_m)) \). Consequently,
\[
\sum_{\sigma \in A} e^{-\theta \cdot D_1(\sigma)} = \psi(|G| - r_0, \theta),
\]
\[
\sum_{\sigma \in B} e^{-\theta \cdot D_2(\sigma)} = \psi(|S| + r_0, \theta) \cdot e^{-\theta(k - 1 - \sum_{m=1}^{l-1} r_m)} \cdot \frac{1}{1 + \cdots + e^{-\theta(|S| + r_0 - 1)}},
\]
where the second equality follows from Theorem EC.1. To complete the proof, it remains to compute
\[
\sum_{\gamma \in \Gamma_R} e^{-\theta \sum_{m \in [M]} D_3(\widehat{G}_m)}. 
\]
Since \( D_3(\widehat{G}_m) \) only involves comparison of products in \( G_m \), we can again split the sum as follows:
\[
\sum_{\gamma \in \Gamma_R} e^{-\theta \sum_{m \in [M]} D_3(\widehat{G}_m)} = \prod_{m=1}^{M} \sum_{\widehat{G}_m \subseteq G_m, |\widehat{G}_m| = r_m} e^{-\theta \cdot D_3(\widehat{G}_m)}
\]
For a given \( m \), using a similar trick, we can write the normalization constant by first conditioning on a subset \( \widehat{G}_m \) as the subset of top \( r_m \) products. This yields
\[
\psi(|G_m|, \theta) = \psi(r_m, \theta) \cdot \psi(|G_m| - r_m, \theta) \cdot \sum_{\widehat{G}_m \subseteq G_m, |\widehat{G}_m| = r_m} e^{-\theta \cdot D_3(\widehat{G}_m)},
\]
which implies that
\[
\sum_{\gamma \in \Gamma} e^{-\theta \sum_{m \in [M]} D_3(\widehat{G}_m)} = \prod_{m=1}^{M} \frac{\psi(|G_m|, \theta)}{\psi(r_m, \theta) \cdot \psi(|G_m| - r_m, \theta)},
\]
and concludes the proof.

**Appendix F: MIP formulation for mixture of Mallows**

We extend the MIP formulation presented in Theorem 2 to the mixture of Mallows model by creating a copy of each variable for each mixture component. The following MIP computes an optimal solution to the unconstrained assortment optimization problem under a mixture of Mallows model:

\[
\begin{align*}
\max_{x^k, \pi^k, y^k, z^k} & \quad \sum_{i, s, k} r_i \cdot p^k_i \cdot \pi^k_{i, s, n} \\
\text{s.t.} & \quad 
\pi^k_{i, s, m+1} = (1 - \gamma_{s, m+1}) \cdot \pi^k_{i, s, m} + y^k_{i, s, m+1}, \quad \forall m < n, \forall i, s \in [m], \forall k \\
& \quad y^k_{i, s, m+1} \leq \gamma_{s-1, m+1} \cdot \pi^k_{i, s-1, m}, \quad \forall m < n, \forall i, s \in [m], \forall k \\
& \quad 0 \leq y^k_{i, s, m+1} \leq \gamma_{s-1, m+1} \cdot \left(1 - x^k_{u^{-1}(m+1)}\right), \quad \forall m < n, \forall i, s \in [m], \forall k \\
& \quad \pi^k_{m+1, s, m+1} = z^k_{s, m+1}, \quad \forall m < n, \forall s \in [m], \forall k \\
& \quad z^k_{s, m+1} \leq \alpha_{s, m+1} \cdot \sum_{i \leq s} \pi^k_{i, \ell, m}, \quad \forall m < n, \forall s \in [m], \forall k \\
& \quad 0 \leq z^k_{s, m+1} \leq \alpha_{s, m+1} \cdot x^k_{w^{-1}(m+1)}, \quad \forall m < n, \forall s \in [m], \forall k
\end{align*}
\]
$\pi_{i,i,i}^k = f_i^k$, \quad \forall k, \forall i \leq w_k(a_q)$

$\pi_{m,s,m}^k = 0$, \quad \forall m, \forall s \geq m + 1, \forall k$

$\pi_{i,s,m}^k = 0$, \quad \forall m, s, \forall i > m, \forall k$

$f_i^k \leq x_{w_k^{-1}(i)}$, \quad \forall k, \forall i \leq w_k(a_q)$

$f_i^k \leq 1 - \sum_{j < i} f_j^k$, \quad \forall k, \forall i \leq w_k(a_q)$

$\sum_{i \leq w_k(a_q)} f_i^k = 1$, \quad \forall k$

$x_q = 1, x_i \in \{0, 1\}, f_i^k \in \{0, 1\}$,

where $w_1, \ldots, w_K$ are the modal rankings of the mixture of Mallows model.

**Appendix G: Collapsed MIP Formulation**

In order to introduce our collapsed formulation, we define two sets of variables, $\pi(s, m)$ and $re(s, m)$. These variables have the following interpretation: $\pi(s, m)$ is the probability of a product being picked in position $s$ after the $m$-th step of Algorithm 1; $re(s, m)$ is the revenue from the product picked in position $s$ after the $m$-th step. With this notation, Algorithm 3 presents the adapted recursion to compute these quantities. The collapsed MIP follows in a very similar way to how Theorem 2 is derived, and we therefore omit its proof. We also assume that $a_1 \in S^*$ for ease of exposition; this assumption can easily be relaxed.

**Algorithm 3** Computing choice probabilities

1: Let $S$ be a general offer set. Without loss of generality, we assume that $a_1 \in S$.
2: Let $\pi(1, 1) = 1$ and $r(1, 1) = r_1$.
3: For $m = 1, \ldots, n - 1$, and $s = 1, \ldots, m + 1$, let

\[
\pi(s, m + 1) = (1 - \gamma_{s,m+1}) \cdot \pi(s, m) + 1[a_{m+1} \notin S] \cdot \gamma_{s-1,m+1} \cdot \pi(s-1, m) \\
+ 1[a_{m+1} \in S] \cdot \alpha_{s,m+1} \cdot \sum_{\ell = s}^{n} \pi(\ell, m),
\]

\[
re(s, m + 1) = (1 - \gamma_{s,m+1}) \cdot re(s, m) + 1[a_{m+1} \notin S] \cdot \gamma_{s-1,m+1} \cdot re(s-1, m) \\
+ 1[a_{m+1} \in S] \cdot r_{m+1} \cdot \alpha_{s,m+1} \cdot \sum_{\ell = s}^{n} \pi(\ell, m).
\]

**Theorem EC.3.** Conditional on $a_1 \in S^*$, the following MIP computes an optimal solution to the unconstrained assortment optimization problem under the Mallows model:

$$\max_{x, \pi, y, z, re, u} \sum_s re_{s,n}$$
The collapsed MIP for the mixture of Mallows model can be obtained similarly.

**Appendix H: Variable Bound Strengthening**

We present the variable bound strengthening used to alleviate the running time dependence on $e^{-\theta}$.

The first step is to multiply each constraint corresponding to the $m$-th step of the dynamic program by both $x_{m+1}$ and $1 - x_{m+1}$, and then to linearize the bilinear terms. More precisely, when the constraint

$$\pi_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot \pi_{s,m} + y_{s,m+1} + z_{s,m+1},$$

is multiplied by $x_{m+1}$, it becomes

$$x_{m+1} \cdot \pi_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot x_{m+1} \cdot \pi_{s,m} + x_{m+1} \cdot y_{s,m+1} + x_{m+1} \cdot z_{s,m+1}.$$  \hspace{1cm} (EC.3)

Recalling that $y_{s,m+1} = \gamma_{s-1,m+1} \cdot (1 - x_{m+1}) \cdot \pi_{s-1,m}$ and $z_{s,m+1} = \alpha_{s,m+1} \cdot x_{m+1} \cdot \sum_{\ell=s}^{n} \pi_{\ell,m}$, by substituting these quantities and noting that $x_{m+1} \cdot (1 - x_{m+1}) = 0$ and $x_{m+1}^2 = x_{m+1}$, the right-hand-side of constraint (EC.4) can be simplified to obtain

$$x_{m+1} \cdot \pi_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot x_{m+1} \cdot \pi_{s,m} + \alpha_{s,m+1} \cdot x_{m+1} \cdot \sum_{\ell=s}^{n} \pi_{\ell,m}.$$  \hspace{1cm} (EC.4)

Similarly, when multiplying by $1 - x_{m+1}$, we have the complementary equation:

$$(1 - x_{m+1}) \cdot \pi_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot (1 - x_{m+1}) \cdot \pi_{s,m} + \gamma_{s-1,m+1} \cdot (1 - x_{m+1}) \cdot \pi_{s-1,m}.$$  

We now linearize this pair of constraints by introducing two variables, $\pi^1$ and $\pi^0$, such that $\pi^1_{s,m+1} = x_{m+1} \cdot \pi_{s,m+1}$ and $\pi^0_{s,m+1} = x_{m+1} \cdot \pi_{s,m}$. More precisely, we replace (EC.3) by the following set of constraints:

$$\pi^1_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot \pi^0_{s,m+1} + \alpha_{s,m+1} \cdot \sum_{\ell=s}^{n} \pi^0_{\ell,m},$$

Electronic copy available at: https://ssrn.com/abstract=3172997
\[
\pi_{s,m+1} - \pi_{s,m+1}^0 = (1 - \gamma_{s,m+1}) \cdot (\pi_{s,m} - \pi_{s,m+1}^0) + \gamma_{s-1,m+1} \cdot (\pi_{s-1,m} - \pi_{s-1,m}^0)
\]
\[
\pi_{s,m+1}^0 \leq \pi_{s,m+1}
\]
\[
\pi_{s,m+1}^1 \leq x_{m+1}
\]
\[
\pi_{s,m+1}^1 \leq \pi_{s,m+1} + x_{m+1} - 1
\]
\[
\pi_{s,m+1}^0 \leq \pi_{s,m}
\]
\[
\pi_{s,m+1}^0 \leq x_{m+1}
\]
\[
\pi_{s,m+1}^0 \leq \pi_{s,m} + x_{m+1} - 1
\]
\[\max_{x, \pi, \pi^0, r, r, r^0} \sum_s r_{s,n} \]
\[\text{s.t.} \quad \pi_{s,m+1}^1 = (1 - \gamma_{s,m+1}) \cdot (\pi_{s,m} - \pi_{s,m+1}^1) + \gamma_{s-1,m+1} \cdot (\pi_{s-1,m} - \pi_{s-1,m}^1), \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1} - \pi_{s,m+1}^1 = (1 - \gamma_{s,m+1}) \cdot (\pi_{s,m} - \pi_{s,m+1}^0) + \gamma_{s-1,m+1} \cdot (\pi_{s-1,m} - \pi_{s-1,m}^0), \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot r_{s,m} + \gamma_{s-1,m+1} \cdot (r_{s-1,m} - r_{s-1,m}), \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1} = (1 - \gamma_{s,m+1}) \cdot (r_{s,m} - r_{s,m}^1) + \gamma_{s-1,m+1} \cdot (r_{s-1,m} - r_{s-1,m}^0), \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^1 \leq \pi_{s,m+1}, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^1 \leq x_{m+1}, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^1 \leq \pi_{s,m+1} + x_{m+1} - 1, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^0 \leq \pi_{s,m}, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^0 \leq x_{m+1}, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,m+1}^0 \leq \pi_{s,m} + x_{m+1} - 1, \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1}^1 \leq r_{s,m+1}, \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1}^1 \leq r_{s,m+1} + x_{m+1} - 1, \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1}^0 \leq r_{s,m}, \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1}^0 \leq x_{m+1}, \quad \forall m < n, \forall s \in [m]
\]
\[r_{s,m+1}^0 \leq r_{s,m} + x_{m+1} - 1, \quad \forall m < n, \forall s \in [m]
\]
\[\pi_{s,1} = 1, \pi_{s,1}^0 = 0, \quad \forall s \geq 2
\]
\[x_1 = 1, x_q = 1, x_i \in \{0, 1\} \quad \forall i.
\]

Note that the number of added constraints is only linear in \( n \). We repeat these operations with the variables \( r e_{s,m} \), which overall leads to the following strengthened MIP:
We implement the additional improvement of fixing in advance decisions regarding the first few products by imposing constraints on the values of $x_2$, $x_3$, and $x_4$.

**Appendix I: Proof of Theorem 5**

**I.1. Instance construction and analysis**

*Construction.* Consider the family of instances parameterized by $n \geq 16$ products, with concentration parameter $\theta = \ln(2)$, central ranking $\omega = a_1 \cdots a_n$, and revenues

$$r_i = \begin{cases} 2^i, & \text{if } i \geq c \\ 0, & \text{otherwise} \end{cases},$$

where $c = \lceil n/2 \rceil$. We also assume that $q = 1$, i.e., the no-purchase is the highest ranked product in $\omega$.

*Analysis.* To establish the theorem, consider the assortment $\hat{S} = \{a_c, \ldots, a_n\}$. While we do not know whether $\hat{S}$ is an optimal assortment, its expected revenue can still be used as a lower bound on the optimal revenue, $\mathcal{R}(S^*)$. For this purpose, we begin by deriving two probabilistic claims, whose proofs are provided in Appendices I.2 and I.3.

**Claim EC.1.** $\mathbb{P}(a_c | \{a_c, a_q\}) \leq 2 \cdot \mathbb{P}(a_c | \hat{S} \cup \{a_q\})$.

**Claim EC.2.** $\mathbb{P}(a_i | \{a_i, a_q\}) \leq 32 \cdot 2^{c-i} \cdot \mathbb{P}(a_c | \{a_c, a_q\})$, for every $i \geq c$.

We proceed by upper-bounding the expected revenue of any single-product assortment, showing in particular that $\mathcal{R}(\{a_i\}) \leq \frac{128}{n} \cdot \mathcal{R}(S^*)$. This inequality clearly holds for $i < c$, since $r_i = 0$ in this regime. When $i \geq c$,

$$\mathcal{R}(\{a_i\}) = r_i \cdot \mathbb{P}(a_c | \{a_i, a_q\})$$

$$\leq 32 \cdot r_i \cdot 2^{c-i} \cdot \mathbb{P}(a_c | \{a_c, a_q\}) \quad \text{(EC.5)}$$

$$\leq 64 \cdot r_i \cdot \mathbb{P}(a_c | \hat{S} \cup \{a_q\}) \quad \text{(EC.6)}$$

$$= \frac{64}{|\hat{S}|} \cdot \mathcal{R}(\hat{S}) \quad \text{(EC.7)}$$

$$\leq \frac{128}{n} \cdot \mathcal{R}(S^*) \quad \text{(EC.8)}.$$  

Here, inequality (EC.5) follows from Claim EC.1 and the observation that $r_c = 2^{c-1} \cdot r_i$. Then, inequality (EC.6) is obtained by combining Claim EC.1 and the observation that $r_c = 2^{c-1} \cdot r_i$. Then, inequality (EC.7) can easily be derived via the relation between the choice probabilities of successive products in $\hat{S}$, stating that $\mathbb{P}(a_{i+1} | \hat{S} \cup \{a_q\}) = \frac{1}{2} \cdot \mathbb{P}(a_{i} | \hat{S} \cup \{a_q\})$ for all $i \geq c$, which we explain in Appendix I.2. Therefore, $r_{i+1} \cdot \mathbb{P}(a_{i+1} | \hat{S} \cup \{a_q\}) = r_i \cdot \mathbb{P}(a_{i} | \hat{S} \cup \{a_q\})$, implying in turn that $\mathcal{R}(\hat{S}) = |\hat{S}| \cdot r_c \cdot \mathbb{P}(a_c | \hat{S} \cup \{a_q\})$. Finally, inequality (EC.8) holds since $|\hat{S}| = n - c + 1 \geq n/2$ and since $\mathcal{R}(\hat{S}) \leq \mathcal{R}(S^*)$.

To conclude the proof, note that for any assortment $S$ and product $a_i \in S$, we have $\mathbb{P}(a_i | S \cup \{a_q\}) \leq \mathbb{P}(a_i | \{a_i, a_q\})$, and consequently,

$$\mathcal{R}(S) \leq \sum_{i \in S} \mathcal{R}(\{a_i\}) \leq \frac{128}{n} \cdot |S| \cdot \mathcal{R}(S^*).$$

As a result, if $|S| < \frac{n}{128} \cdot n$, we necessarily have $\mathcal{R}(S) < \alpha \cdot \mathcal{R}(S^*)$, and the theorem follows.
I.2. Proof of Claim EC.1

We first observe that the probability of \( a_c \) to be chosen when \( \{a_c, a_q\} \) is the offer set is smaller than the probability of choosing one of the products in \( \hat{S} \) when \( \hat{S} \cup \{a_q\} \) is the offer set. Indeed, in any ranking where \( a_c \) is preferred to \( a_q \), some product of \( \hat{S} \) is preferred to \( a_q \) as well, since \( a_c \in \hat{S} \). Consequently,

\[
\mathbb{P}(a_c \mid \{a_c, a_q\}) \leq \sum_{j=c}^{n} \mathbb{P}(a_j \mid \hat{S} \cup \{a_q\}) = \mathbb{P}(a_c \mid \hat{S} \cup \{a_q\}) \leq 2 \cdot \mathbb{P}(a_c \mid \hat{S} \cup \{a_q\}).
\]

To understand the equality above, note that Theorem EC.2 implies \( \mathbb{P}(a_{i+1} \mid \hat{S} \cup \{a_q\}) = e^{-\theta} \cdot \mathbb{P}(a_i \mid \hat{S} \cup \{a_q\}) \), for all \( i \geq c \). Indeed, upon inspection of the closed-form expression in this theorem, if an assortment contains two consecutively-ranked products (according to \( \omega \)), their choice probabilities differ by exactly \( e^{-\theta} \).

I.3. Proof of Claim EC.2

We begin by providing an explicit expression for \( \mathbb{P}(a_i \mid \{a_i, a_q\}) \) for \( i \geq c \). To this end, Marden (1995, Thm. 6.3) showed that \( \mathbb{P}(a_i \mid \{a_i, a_q\}) = h(j - i + 1; \theta) - h(j - i; \theta) \) for \( i < j \), where \( h(k; \theta) = \frac{k}{1 - e^{-k\theta}} \). In our case, since the no-purchase option is top-ranked (i.e., \( q = 1 \)), it follows that for all \( i \geq 2 \),

\[
\mathbb{P}(a_i \mid \{a_i, a_q\}) = 1 - \frac{i - 1}{1 - e^{-i\theta}} + \frac{i - 1}{1 - e^{-(i-1)\theta}}
= i \cdot \left( \frac{1}{1 - e^{-i\theta}} - \frac{1}{1 - e^{-(i-1)\theta}} \right) - \frac{e^{-(i-1)\theta}}{1 - e^{-(i-1)\theta}}
= i \cdot \frac{e^{-(i-1)\theta} - e^{-i\theta}}{(1 - e^{-i\theta}) \cdot (1 - e^{-(i-1)\theta})} - \frac{e^{-(i-1)\theta}}{1 - e^{-(i-1)\theta}}.
\]

We mention in passing that this expression can alternatively be derived from Theorem EC.2. Therefore, for all \( i \geq c \),

\[
\mathbb{P}(a_i \mid \{a_i, a_q\}) \leq \frac{n \cdot e^{-(i-1)\theta}}{(1 - e^{-\theta})^2} = 4n \cdot 2^{-(i-1)}.
\]

On the other hand,

\[
\mathbb{P}(a_c \mid \{a_c, a_q\}) \geq c \cdot e^{-(c-1)\theta} \cdot (1 - e^{-\theta}) - \frac{e^{-(c-1)\theta}}{1 - e^{-\theta}}
= \left( \frac{c}{2} - 2 \right) \cdot 2^{-(c-1)}
= \left( \frac{\lfloor n/2 \rfloor}{2} - 2 \right) \cdot 2^{-(c-1)}
\geq \frac{n}{8} \cdot 2^{-(c-1)},
\]

where the last holds since \( n \geq 16 \). Consequently, we indeed get

\[
\frac{\mathbb{P}(a_c \mid \{a_c, a_q\})}{\mathbb{P}(a_i \mid \{a_i, a_q\})} \geq \frac{1}{32} \cdot 2^{-(c-1)}.
\]
Appendix J: Quasi-PTAS: Extension to Multiple Segments

In what follows, we derive Theorem 6 by extending the quasi-PTAS presented in Section 8 to a mixture of Mallows model with an arbitrary number of segments.

J.1. Definitions and notation

Let $S^*$ be some fixed optimal assortment with respect to the mixture of Mallows model over the segments $(\omega_1, \theta_1), \ldots, (\omega_K, \theta_K)$, where the probabilities that a random customer belongs to each of these segments are denoted by $p_1, \ldots, p_K$, respectively. For purposes of analysis, we introduce a number of definitions along with some useful notation. As in the case of a single segment, we make use of the parameters $\mathcal{P} = \{0, \ldots, \lceil \log_{1+\epsilon} \Delta \rceil \}$ and $\mathcal{L} = \{0, \ldots, \lceil \log_{1+\epsilon} (\frac{n\Delta}{r}) \rceil \}$, where $\Delta = \frac{\max}{\min}$ stands for the ratio between the maximal and minimal prices of any product.

J.1.1. Configuration subsets of $S^*$. We begin by constructing a family of pairwise-disjoint subsets of the optimal assortment $S^*$, according to certain joint configurations of its products. To this end, for every $k \in [K]$, let $d_k^*$ be the product in $S^*$ whose rank in $\omega_k$ is minimal. For every integer $p \in \mathcal{P}$ and for every vector $\ell = (\ell_1, \ldots, \ell_K) \in \mathcal{L}^K$, we define $\mathcal{I}_\ell^p$ to be the set of products $a \in S^*$ for which:

1. Choice probabilities: $\mathbb{P}_k(a|S^*) \in \left( \frac{p_k(d_k^*|S^*)}{1+\epsilon}, \frac{p_k(a|S^*)}{1+\epsilon} \right]$, for every $k \in [K]$.
2. Price $r_\ell \in [(1+\epsilon)^p \cdot r_{\min}, (1+\epsilon)^{p+1} \cdot r_{\min}]$.

In item 1 above, $\mathbb{P}_k(\cdot)$ denotes choice probabilities with respect to the segment $(\omega_k, \theta_k)$. It is easy to verify that the subsets $\{\mathcal{I}_\ell^p\}_{p, \ell}$ are pairwise-disjoint. However, there may be products $a \in S^*$ that do not belong in any of these subsets, corresponding to cases where $\mathbb{P}_k(a|S^*) < \mathbb{P}_k(d_k^*|S^*) \cdot (1 + \epsilon)^{-\left(\lceil \log_{1+\epsilon} (\frac{n\Delta}{r}) \rceil + 1\right)}$ for some segment $k$.

J.1.2. Creating blocks. Focusing on a single subset, we say that $\mathcal{I}_\ell^p$ is big when $|\mathcal{I}_\ell^p| \geq K/\epsilon$; otherwise, this subset is said to be small. For every segment $k \in [K]$, we partition every big subset $\mathcal{I}_\ell^p$ into a collection of $O(K/\epsilon)$ blocks, $\mathcal{B}_{\ell,k,1}^p, \ldots, \mathcal{B}_{\ell,k,M_{\ell,k}^p}^p$, defined as follows. Letting $\beta_{\ell,k}^p = \lceil \epsilon \cdot |\mathcal{I}_\ell^p|/K \rceil \geq 1$, the first block $\mathcal{B}_{\ell,k,1}^p$ consists of the $\beta_{\ell,k}^p$ products in $\mathcal{I}_\ell^p$ whose $\omega_k(\cdot)$-value is maximal, i.e., the $\beta_{\ell,k}^p$ least preferred ones in terms of the ranking $\omega_k$. Then, out of the remaining products in $\mathcal{I}_\ell^p$, the second block $\mathcal{B}_{\ell,k,2}^p$ consists of the next $\beta_{\ell,k}^p$ products with largest $\omega_k(\cdot)$-values. So on and so forth, up until the last block $\mathcal{B}_{\ell,k,M_{\ell,k}^p}^p$, that may consist of fewer than $\beta_{\ell,k}^p$ products when $|\mathcal{I}_\ell^p|$ does not divide by $\beta_{\ell,k}^p$. Within each block $\mathcal{B}_{\ell,k,m}^p$, we refer to the most preferred and least preferred products as $\text{high}[\mathcal{B}_{\ell,k,m}^p]$ and $\text{low}[\mathcal{B}_{\ell,k,m}^p]$, respectively. It is important to emphasize that we obtain $K$ different partitions of $\mathcal{I}_\ell^p$, one for each segment.
J.1.3. Product elimination. Finally, for simplicity of presentation, we create a modified assortment \( S^- \) by carefully deleting certain products from \( S^* \). Specifically, \( S^- \) is obtained by eliminating from the optimal assortment \( S^* \) all products in the blocks \( B_{\ell,k,M_{\ell}^{p}}^p \) and \( B_{\ell,k,M_{\ell}^{p-1}}^p \), over all big subsets \( \mathcal{I}_{\ell}^p \) and segments \( k \in [K] \). Accordingly, for every original block \( B_{\ell,k,m}^p \), we designate by \( B_{\ell,k,m}^- = S^- \cap B_{\ell,k,m}^p \) the corresponding subset of products that survived this eliminate step. Once again, we mention in passing that the preceding discussion is intended for purposes of analysis only, and does not involve any algorithmic actions.

J.2. The algorithm

Our algorithm constructs an assortment \( T \) as follows:

1. Guessing subset type: For any integer \( p \in \mathcal{P} \) and for every vector \( \ell \in \mathcal{L}^{K} \), we guess whether the subset \( \mathcal{I}_{\ell}^p \) is big or small.

2. Handling small subsets: Here, we simply guess all products in \( \mathcal{I}_{\ell}^p \) and include them in \( T \).

3. Handling big subsets:
   (a) We initially guess the number of blocks \( M_{\ell}^p \).
   (b) For every segment \( k \in [K] \) and for every block index \( m \in [M_{\ell}^p - 2] \), we guess the block size \( \beta_{\ell,k,m}^p = |B_{\ell,k,m}^-| \) as well as the products high\([B_{\ell,k,m}^-]\) and low\([B_{\ell,k,m}^-]\) whenever \( \beta_{\ell,k,m}^p \geq 1 \).
   (c) Finally, let \( \bar{B}^p \) be the collection of products \( a \in [n] \) with price \( r_a \in [(1 + \epsilon)^p \cdot r_{\min}, (1 + \epsilon)^{p+1} \cdot r_{\min}] \). We include in \( T \) a set of products \( T_{\ell}^p \) picked out of \( \bar{B}^p \) with the following properties:
      i. For every segment \( k \in [K] \) and for every block index \( m \in [M_{\ell}^p - 2] \), there are precisely \( \beta_{\ell,k,m}^p \) products in \( T_{\ell}^p \) with rank \( \omega_k(a) \in [\omega_k(\text{high}[B_{\ell,k,m}^-]), \omega_k(\text{low}[B_{\ell,k,m}^-])] \).
      ii. For every segment \( k \in [K] \), we have \( \omega_k(a) \in [\omega_k(\text{high}[B_{\ell,k,M_{\ell}^{p-1}}]), \omega_k(\text{low}[B_{\ell,k,1}])] \) for all products \( a \in T_{\ell}^p \).
      iii. For every segment \( k \in [K] \), we have \( \text{low}[B_{\ell,k,1}^-] \subseteq T_{\ell}^p \).

Following this description, a number of comments are in place. First, since \( B_{\ell,k,m}^- \) is a subset of \( \bar{B}^p \) that satisfies properties (i)-(iii) for every combination of \( \ell, k, \) and \( m \), a set \( T_{\ell}^p \subseteq \bar{B}^p \) with the required properties indeed exists. Moreover, the rank condition (ii) ensures that the subsets \( \{T_{\ell}^p\}_\ell \) are pairwise-disjoints. We proceed by showing how to implement step 3(c) for given \( \ell \) and \( p \) values by means of dynamic programming.

Implementing step 3(c). For convenience of notation, we designate the collection of products in \( \bar{B}^p \) by \( b_1, \ldots, b_{|\bar{B}^p|} \). Each state of our dynamic program corresponds to a pair \((i, \beta)\), where \( i \in [|\bar{B}^p|] \) and \( \beta \) is an integer-valued vector, with a separate coordinate \( \beta_{k,m} \in \{0, \ldots, |\bar{B}^p|\} \) for every segment \( k \in [K] \) and for every block index \( m \in [M_{\ell}^p - 2] \). With these states, the value function \( V(i, \beta) \) simply indicates whether there exists a subset of the products \( b_1, \ldots, b_i \) in which, for every \( k \) and \( m \), there are exactly \( \beta_{k,m} \) products with \( \omega_k\)-rank in \([\omega_k(\text{high}[B_{\ell,k,m}^-]), \omega_k(\text{low}[B_{\ell,k,m}^-])]\). This function is recursively defined as follows:
**Initialization.** Clearly, when \( i = 1 \), there are only two feasible subsets, depending on whether we pick \( b_i \) or not. Therefore, \( V(1, 0) = \text{TRUE} \), \( V(1, \beta^1) = \text{TRUE} \), and \( V(1, \beta) = \text{FALSE} \) for any other vector, where \( \beta^1 \) corresponds to the cardinalities resulting from picking \( b_1 \), i.e.,

\[
\beta^1_{k,m} = \begin{cases} 
1, & \text{if } \omega_k(b_1) \in [\omega_k(\text{high}[B^{-p}_{\ell,k,m}]), \omega_k(\text{low}[B^{-p}_{\ell,k,m}])] \\
0, & \text{otherwise}
\end{cases}
\]

**General step.** For \( i \geq 2 \), we begin by defining a vector \( \beta^i \) that captures the cardinalities resulting from picking \( b_i \), i.e.,

\[
\beta^i_{k,m} = \begin{cases} 
1, & \text{if } \omega_k(b_i) \in [\omega_k(\text{high}[B^{-p}_{\ell,k,m}]), \omega_k(\text{low}[B^{-p}_{\ell,k,m}])] \\
0, & \text{otherwise}
\end{cases}
\]

With this notation, when \( b_i = \text{low}[B^{-p}_{\ell,k,1}] \) for some segment \( k \in [K] \), we necessarily have to pick \( b_i \), in which case \( V(i, \beta) = V(i - 1, \beta - b^i) \). In the opposite scenario, where \( b_i \neq \text{low}[B^{-p}_{\ell,k,1}] \) for all \( k \in [K] \), this product can either be picked or not, implying that \( V(i, \beta) = V(i - 1, \beta) \lor V(i - 1, \beta - b^i) \).

As previously mentioned, a set \( T^p_l \subseteq \tilde{B}^p \) that satisfies properties (i)-(iii) indeed exists, and therefore, \( V(|\tilde{B}^p|, \beta^{-p}) = \text{TRUE} \). Backtracking the choices made by the dynamic program allows us to output a subset of products satisfying the required properties.

### J.3. Analysis

The remainder of this section is devoted to proving the specifics of Theorem 6. We begin by discussing the running time of our algorithm and then turn to proving its performance guarantee.

**Running time.** The number of guesses required for step 1 is \( 2^{O(|L|K^2/\epsilon^4)} \). In addition, to implement step 2, since each small subset consists of at most \( K/\epsilon \) products, the total number of guesses for each such subset is \( O(n^{O(K^2/\epsilon^4)}) \). To implement step 3, for every big subset we make use of \( O(K/\epsilon) \) guesses for the number of blocks \( M^p_x \), \( O(n) \) guesses for each of the block sizes \( \beta^{-p}_{\ell,k,m} \), and \( O(n) \) guesses for each of the products \( \text{high}[B^{-p}_{\ell,k,m}] \) and \( \text{low}[B^{-p}_{\ell,k,m}] \) within a given block. Consequently, for each of the big subsets, there are \( O(n^{O(K^2/\epsilon^4)}) \) guesses. Finally, the dynamic program required for step 3(c) is solvable in \( O(n^{O(K^2/\epsilon^4)}) \) time. Therefore, the total running time of our algorithm is

\[
O_t \left( n^{O(|L|K^2/\epsilon^4)} \right) = O_t \left( n^{O(K^2/\epsilon^4)} \right) = O_t \left( n^{O_K \cdot (K^2/\epsilon^4)} \right) = O_t \left( n^{O_K \cdot (K^2/\epsilon^4)} \right) = O_t \left( n^{O_K \cdot (K^2/\epsilon^4)} \right)
\]

**Performance guarantee.** We now turn our attention to proving that the expected revenue of the resulting assortment \( T \) is near-optimal. For this purpose, Lemma EC.1 shows that, for every segment \( (\omega_k, \theta_k) \), there exists an up-mapping from \( T \cap \omega_k \) to \( S^* \cap \omega_k \) that preserves certain product-specific guesses as fixed-points; the proof is omitted, due to being nearly-identical to that of Lemma 2 in the single-segment case. This claim allows us to subsequently argue in Lemma EC.2 that \( R(T) = (1 - O(\epsilon)) \cdot R(S^*) \).
**Lemma EC.1.** For every segment \( k \in [K] \), there exists an up-mapping \( \psi : T \to S^* \) that satisfies the following properties for every integer \( p \in \mathcal{P} \) and every vector \( \ell \in \mathcal{L}^K \):
1. If \( I^p_\ell \) is small, then every product in \( I^p_\ell \) is a fixed-point of \( \psi_k \).
2. If \( I^p_\ell \) is big, then \( \text{low}[\beta_{\ell,k,1}^-] \) is a fixed-point of \( \psi_k \).

**Lemma EC.2.** \( \mathcal{R}(T) \geq (1 - 8\epsilon) \cdot \mathcal{R}(S^*) \).

**Proof.** To derive the desired claim, recall that the expected revenue function is given by \( \mathcal{R}(S) = \sum_{k \in [K]} p_k \cdot \mathcal{R}_k(S) \), where \( \mathcal{R}_k(\cdot) \) is the revenue function restricted to segment \( k \), i.e., conditional on knowing that the customer’s choices are determined via the single-segment Mallows model \((\omega_k, \theta_k)\). Consequently, our proof shows that the assortment \( T \) is near-optimal on a term-by-term basis, meaning that \( \mathcal{R}_k(T) \geq (1 - 8\epsilon) \cdot \mathcal{R}_k(S^*) \) for every segment \( k \in [K] \). The arguments are very similar to the single segment case, with a number of small differences due to the product elimination step.

For a single segment \( k \in [K] \), using the same analysis as in the single segment case, we have that for \( \epsilon \in (0, 1/2] \),

\[
\mathcal{R}_k(S^*) \leq (1 + 2\epsilon) \cdot \sum_{(l,p) \in \mathcal{L} \times \mathcal{P}} \sum_{a \in I^p_{\ell}} r_a \cdot \mathbb{P}_k(a | S^*) . \tag{EC.9}
\]

Our proof proceeds by relating the right-hand-side of this upper bound to the conditional expected revenue \( \mathcal{R}_k(T) \) of the assortment \( T \). For this purpose, the latter quantity can be decomposed into the revenue contributions of our choices for small and big subsets:

\[
\mathcal{R}_k(T) = \sum_{(l,p) : \ell \in I^p_{\ell} \small} \sum_{a \in I^p_{\ell}} r_a \cdot \mathbb{P}_k(a | T) + \sum_{(l,p) : \ell \in I^p_{\ell} \big} \sum_{a \in I^p_{\ell}} r_a \cdot \mathbb{P}_k(a | T) . \tag{EC.10}
\]

To obtain a lower bound on the above expression, let \( \psi_k : T \to S^* \) be an up-mapping that satisfies the structural properties mentioned in Lemma EC.1. With this ingredient in place, property 1 states that every product in a small subset \( I^p_{\ell} \) is a fixed-point of \( \psi_k \). Thus, by Claim 5, it follows that \( \mathbb{P}_k(a | T) \geq \mathbb{P}_k(a | S^*) \) for every such product \( a \), and we have

\[
(I) \geq \sum_{(l,p) : \ell \in I^p_{\ell} \small} \sum_{a \in I^p_{\ell}} r_a \cdot \mathbb{P}_k(a | S^*) .
\]

In the opposite case, where \( I^p_{\ell} \) is a big subset, the inner summation in (II) can be bounded by observing that

\[
\sum_{a \in I^p_{\ell}} r_a \cdot \mathbb{P}_k(a | T) \geq (1 + \epsilon)^p \cdot r_{\text{min}} \cdot \sum_{a \in I^p_{\ell}} \mathbb{P}_k(a | T) \\
\geq (1 + \epsilon)^p \cdot r_{\text{min}} \cdot \mathbb{P}(\text{low}[\beta_{\ell,k,1}^-] | S^*) \cdot \sum_{m \in [M^p_{\ell} - 2]} \beta_{\ell,k,m} . \tag{EC.11}
\]
Here, inequality (EC.11) holds since, by property (i) of step 3c, for every \( m \in [M^*_t - 2] \) there are \( \beta^{-p}_{t,k,m} \) products in \( T_t^p \) with rank \( \omega_k(a) \in [\omega_k(\text{high}[B_{t,k,m}])], \omega_k(\text{low}[B_{t,k,m}]) \). For each such product, since its rank is at most \( \omega_k(\text{low}[B_{t,k,m}]) \), we have \( \mathbb{P}_k(a|T) \geq \mathbb{P}_k(\text{low}[B_{t,k,1}]|T) \) by Claim 5, since \( \text{low}[B_{t,k,1}] \) is guaranteed to be a fixed-point of \( \psi_k \) by Lemma EC.1. To justify inequality (EC.12), note that \( \sum_{m \in [M^*_t - 2]} \beta^{-p}_{t,k,m} \) is precisely the total size of the residual subset of products \( B_{t,k,1}, \ldots, B_{t,k,M^*_t - 2} \) that survived the elimination step, which is at least \( |I^*_t| - 2 \cdot K \cdot \beta^{-p}_{t,k} = |I^*_t| - 2 \cdot K \cdot [\epsilon \cdot |I^*_t|/K] \geq (1 - 4\epsilon) \cdot |I^*_t| \), since \( |I^*_t| \geq K/\epsilon \).

To conclude the proof, we plug the lower bounds on (I) and (II) into equation (EC.10), and get

\[
\mathcal{R}_k(T) \geq (1 - 6\epsilon) \cdot \sum_{(\ell,p) \in \mathcal{E} \times \mathcal{P}} \sum_{a \in T_t^p} r_a \cdot \mathbb{P}_k(a|S^*)
\]

\[
\geq \frac{1 - 6\epsilon}{1 + 2\epsilon} \cdot \mathcal{R}_k(S^*)
\]

\[
\geq (1 - 8\epsilon) \cdot \mathcal{R}_k(S^*)
\]

where the second equality follows from the upper bound in inequality (EC.9). \( \Box \)

**Appendix K: Structure of Optimal Assortments under a Single Mallows Model**

In what follows, we provide an example to show that the optimal assortment under a single Mallows model does not seem to exhibit an intuitive structure. Specifically, suppose that products are ordered in the modal ranking by decreasing price, i.e., \( 1 > 2 > \cdots > q \) and \( r_1 > r_2 > \cdots > 0 \). In this situation, one may suspect that the optimal assortment consists of the top \( k \) products, for some \( k \). However, we show this is not the case. In particular, consider the following instance with \( e^{-\theta} = 0.3 \) and:

- products: \( 1 > 2 > 3 > q \)
- prices: \( 1 = 0.9 > 0.89 > 0 \)

Table EC.5 reports the expected revenues for all non-empty assortments. The optimal assortment is \( S^* = \{1, 3\} \), which does not consist of the top \( k \) products for any \( k \). Moreover, this assortment is also not nested by price.
<table>
<thead>
<tr>
<th>Assortment</th>
<th>Expected revenue</th>
</tr>
</thead>
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<tr>
<td>{1,2,3}</td>
<td>0.952</td>
</tr>
<tr>
<td>{2,3}</td>
<td>0.839</td>
</tr>
<tr>
<td>{1,3}</td>
<td>0.955</td>
</tr>
<tr>
<td>{1,2}</td>
<td>0.954</td>
</tr>
<tr>
<td>{1}</td>
<td>0.949</td>
</tr>
<tr>
<td>{2}</td>
<td>0.796</td>
</tr>
<tr>
<td>{3}</td>
<td>0.684</td>
</tr>
</tbody>
</table>

**Table EC.5** Expected revenues of all non-empty assortments.